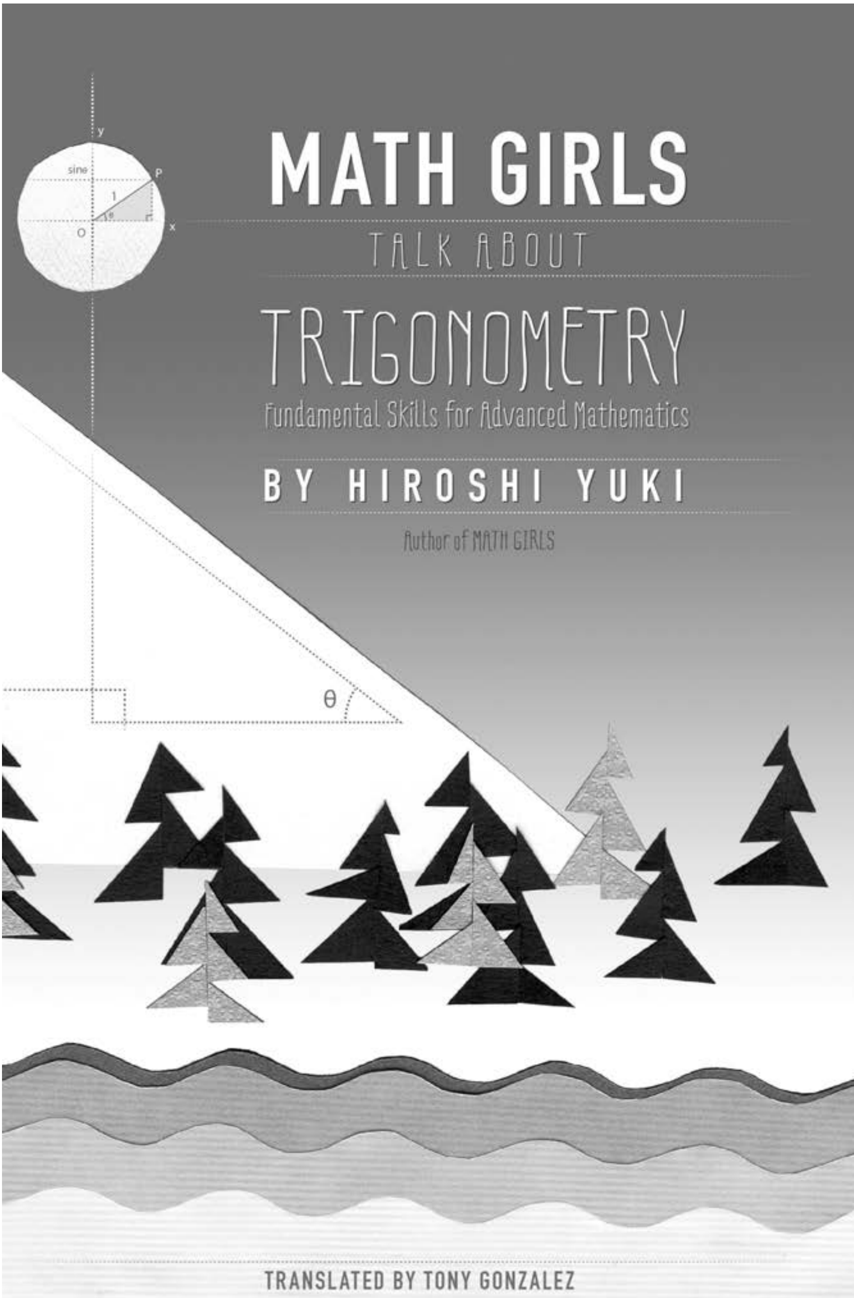

MATH GIRLS

TALK ABOUT

TRIGONOMETRY

Fundamental Skills for Advanced Mathematics

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TRANSLATED BY TONY GONZALEZ

MATH GIRLS TALK ABOUT TRIGONOMETRY

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Math Girls Talk About
Trigonometry

Contents

Chapter 1

Round Triangles

> "Names describe forms. Forms represent essence."

1.1 Starting the Journey

In the library after school one day, I found Tetra furiously scribbling equations in her notebook.

Me "Hey, Tetra. Doing some math?"

Tetra "Yeah. I never thought it would happen, but after starting to study with you guys, I'm actually learning to like this stuff!"

Me "So what are you working on now?"

Tetra "Something completely new! Trigonometry!"

Me "Sines and cosines and all that, huh."

Tetra "Yeah..."

Tetra's face darkens.

Me "What's wrong?"

Tetra "Well, I was hoping this would give me more fun things to talk about with you and Miruka, but it's harder than I expected."

Me "It can be confusing at first, but it's not so bad once you get used to it."

Tetra "At first I thought it was just about playing with triangles, but it looks like there's all kinds of other stuff too. Just what *is* trigonometry?"

Me "I don't think there's a simple answer to that. How about we talk about it some, and see what we find?"

Tetra "That would be great!"

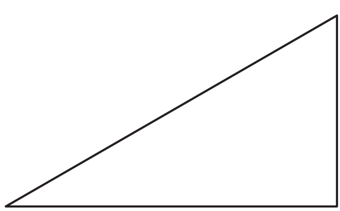
1.2 Right Triangles

Me "I don't know how deep you've gotten so far, so I'll start with the absolute basics."

Tetra "Perfect."

Me "Okay, start by drawing a right triangle."

Tetra "Like this?"

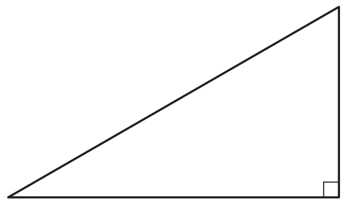


Me "Well, that *looks* like a right triangle . . . "

Tetra "Did I do something wrong?"

Me "When you draw a right triangle, it's best to put a box on the right angle."

Tetra "Oh, I've seen that! Like this, right?"



A right triangle with the right angle indicated

Me "Exactly. That makes sure the reader knows what you're trying to show."

Tetra "Got it."

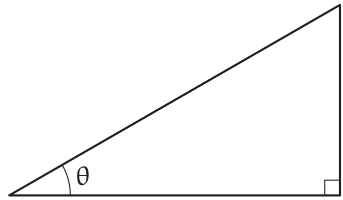
Tetra pulls out her "secret notebook" and makes a quick note.

1.3 Naming Angles

Me "Some more basics. A triangle has three angles, and in a right triangle one of those angles is 90°."

Tetra "That's the right angle that makes right triangles right. Right?"

Me "Uh . . . correct. So, anyway, there are two other angles. We're going to name one of them theta."



We name one of the angles θ

Tetra "Theta . . . That's how you read this Greek letter?"

Me "It is. Angles are often named using Greek letters. That's just a convention, though. We don't have to name them like that, if you don't want to."

Tetra "No, that's fine. Let's go with the Greek."

Me "Just checking. Math uses lots of symbols to name things, and I know that puts some people off."

Tetra "Well, to be honest I still get a little freaked out when there's a whole bunch of them. It starts to feel like it's all too much to digest."

Me "Everybody feels that way at first. Just take it all in at your own pace, and you'll be fine."

Tetra "I dunno . . . My pace can be pretty slow."

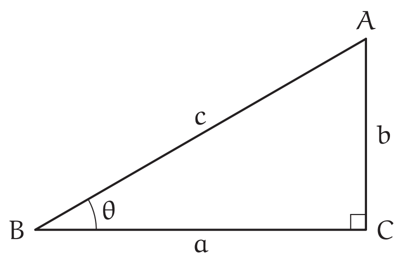
Me "That's okay, the problem isn't going anywhere. It's better to go slow, making sure you get everything, than to race through a bunch of stuff you don't understand. Take time to get to know each symbol, and become friends with it."

Tetra "I guess if I'm going to master trig, I'll be making a lot of Greek friends!"

Tetra's eyes widens as she smiles.

1.4 Naming Vertices and Sides

Me "Since we're naming things, let's name the vertices and sides of this triangle, too."



Triangle ABC

Tetra "So you named the vertices A, B, and C, right?"

Me "Using capital letters, which is another convention. We can string those letters together and call the triangle as a whole $\triangle ABC$. You usually label parts of a triangle in counter-clockwise order, but—"

Tetra "—but that's just another convention, so you don't *have* to, right?"

Me "Exactly. But there's one rule you've gotta stick to: when you write about a shape, be sure to use the same names that are in the diagrams."

Tetra "Meaning?"

Me "Like if you write something about $\triangle ABC$, be absolutely sure the vertices of that triangle are labelled A, B, C in any diagrams."

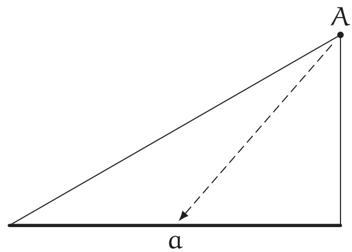
Tetra "Got it. Here's A, here's B, and here's C. Looks good!"

Me "When you name sides, you'll usually use lowercase letters. That makes it easier to distinguish between sides and vertices."

Tetra “Oh, okay.”

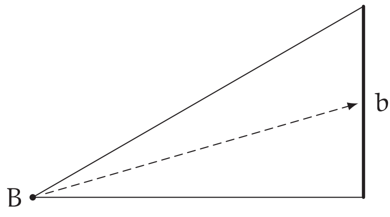
Me “Also, you normally use the lowercase version of the letter you used to name the vertex that’s across from the side you’re naming.”

Tetra “So we have side lowercase-a across from vertex uppercase-A.”

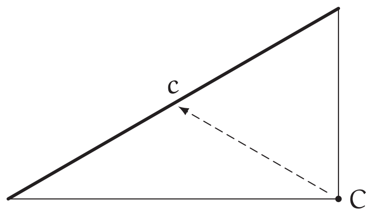


Name sides using the lowercase version of the letter used to name their opposite vertices

Me “And side lowercase-b across from vertex uppercase-B—”



Tetra “—and side lowercase-c across from vertex uppercase-C.”



Me "One thing to be careful about: when you want to talk about the *length* of a side, sometimes it's clearer to use the names of that side's vertices with a bar over them. So if I write this . . . "

$$\overline{AB}$$

Me " . . . that means 'the length of the side formed by vertices A and B.' "

Tetra "So a lowercase letter is the name of a side, and a pair of vertices with a line over them is the length of a side."

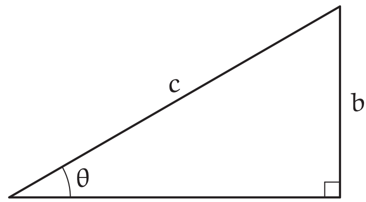
Me "Right. Again, you don't *have* to do things this way. The math will all work the same, no matter what you name things. But this is how people have been doing it for a while now, so unless you have a good reason to do things differently your math will be more legible if you follow their lead."

Tetra "Gotcha!"

1.5 The Sine Function

Me "Let's take a closer look at $\angle\theta$ and the sides b and c in this triangle."

Tetra "Okay."

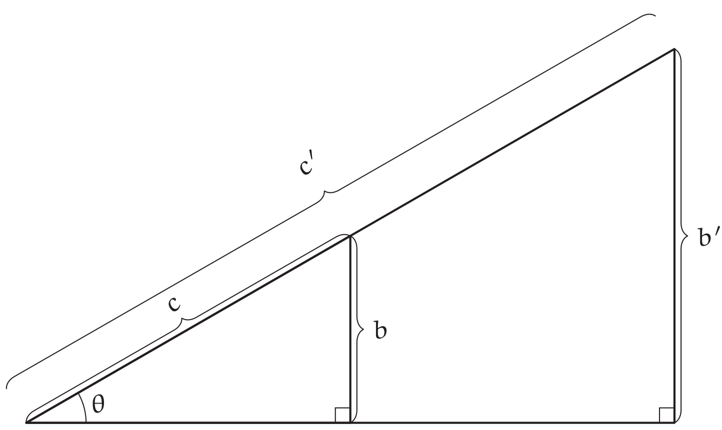


Tetra takes me literally, leaning in to take a closer look at the diagram. She points at the angle and sides, and I hear her muttering their names.

Me "We want to think about the relationship between the size of $\angle\theta$ and the length of those sides."

Tetra "They're related?"

Me "Sure. Think about what would happen if you made side c longer, *without changing the angle of* θ. For example, if we made side c twice as long, our right triangle would look like this."



Tetra "Side c' is the doubled side c, right?"

Me "It is. And if we want to do that to side c and still have a right triangle, then we need to stretch side b, too. Side b' here is that stretched side b."

Tetra "Makes sense."

Me "In fact, if we want to double the length of side b, we have to double the length of side c as well. And if we triple one we have to triple the other, and so on."

Tetra "So the length of sides b and c are in proportion."

Me "Exactly! In other words, if the measure of $\angle\theta$ remains constant, then the ratio of sides b and c must be a constant, too."

Tetra "A constant ratio . . . ?"

Me "Put another way, if $\angle\theta$ remains constant, then the value of the fraction $\frac{b}{c}$ remains constant."

Tetra "So like if you double or triple the c in the denominator, you have to double or triple the b in the numerator, too."

Me "That's right."

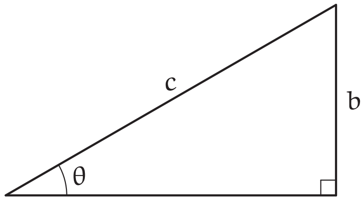
Tetra "Okay, I get that. But this all sounds more like geometry. Does it have anything to do with trigonometry?"

Me "This *is* trigonometry."

Tetra "Huh?"

Me "Look at it this way. Here's what we've been saying."

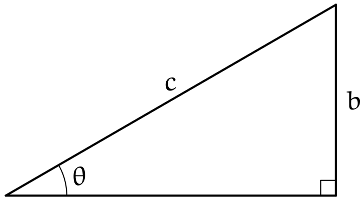
If the measure of an angle $\angle\theta$ in a right triangle is constant, then the value of the fraction $\frac{b}{c}$ is constant, too.



Tetra “Sure, I get that.”

Me “But here’s another way to look at it.”

Determining the measure of an angle $\angle\theta$ in a right triangle also determines the value of the fraction $\frac{b}{c}$.



Tetra “Let me make sure I’ve got this straight. We want θ to be a certain angle, and we want the triangle to be a right triangle. Side lengths b and c are proportional, which means their ratio is some constant value. So when we set θ to some value, that determines what the value of $\frac{b}{c}$ will be.”

Me “An excellent summary.”

Tetra "Good. I'm not sure how you would find the angle, but I see how setting θ to something would set the value of $\frac{b}{c}$, too."

Me "Then how about we give a name to that value of $\frac{b}{c}$ that you get from setting θ. Turns out there's already a very good one, so we'll use that and call it $\sin\theta$."

Tetra's eyes widen, and she grabs my arm.

Tetra "Sine!? Did you just say sine, like the trig function sine? That's all it is?"

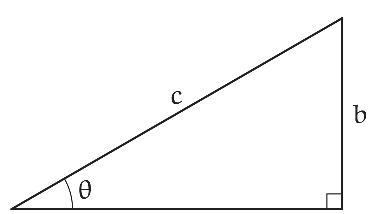
Me "What do you mean, 'that's all'?"

Tetra "You're saying that $\sin\theta$ is just the value of this ratio $\frac{b}{c}$?"

Me "That's all it is. Well, let me take that back. So far we've only defined it using right triangles, so that's all it is for a θ in the range $0° < \theta < 90°$. But it's correct to say that $\sin\theta$ is equal to the value of this ratio $\frac{b}{c}$."

Definition of $\sin\theta$ as a ratio of two sides in a right triangle ($0° < \theta < 90°$)

$$\sin\theta = \frac{b}{c}$$



Tetra writes this in her notebook, whispering "Wow . . . wow . . ." as she does.

1.6 How to Remember Sine

Tetra "I *must* have read this in my book, right?"

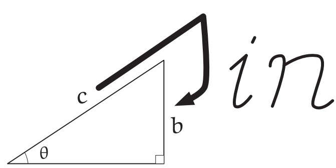
Me "I'd think so. I can't imagine teaching trig functions without mentioning ratios of triangle side lengths."

Tetra "I must have gotten lost in all the symbols."

Me "Yeah? It doesn't seem like there's that many . . . "

Tetra "It's not so much that there's a whole lot of them. The problem is that there's three sides, and I can never remember which ones go in the numerator and the denominator."

Me "Oh, well here's a good trick for that. Using this triangle, write a cursive, lower-case 's' and say 'c divides b.' The 's' stands for 'sine,' and the 'c divides b' means you want the fraction $\frac{b}{c}$. Like this."[1]



How to remember $\sin\theta$

[1] This is a mnemonic device commonly used in Japan. Readers unfamiliar with cursive script may have better success using mnemonics devised from the abbreviations SOH (sine = opposite ÷ hypotenuse), CAH (cosine = adjacent ÷ hypotenuse), and TOA (tangent = opposite ÷ adjacent), either simply as syllables (SOH-CAH-TOA) or by expanding those letters into a sentence, such as "Socks On Hard Concrete Always Hurts—Try Other Alternatives."

Tetra “How do I remember where the right angle should be?”

Me “Not a problem—you don’t have to worry about where the right angle is, just where the angle you’re taking the sine of is. You can just mentally move the triangle around so that θ is on the left.”

Tetra “Oh, okay.”

Me “Anyway, regardless of how you remember it, the most important thing is understanding what it is the sine function is using, and what it’s finding.”

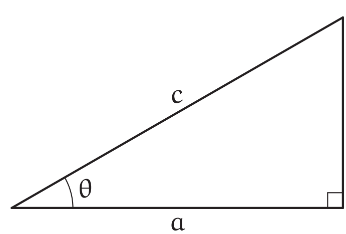
Tetra “Maybe I don’t quite understand that yet.”

Me “Sure you do. Like we said, once θ is set, that determines $\frac{b}{c}$. In other words, sine is a function that finds $\frac{b}{c}$ from $\angle\theta$.”

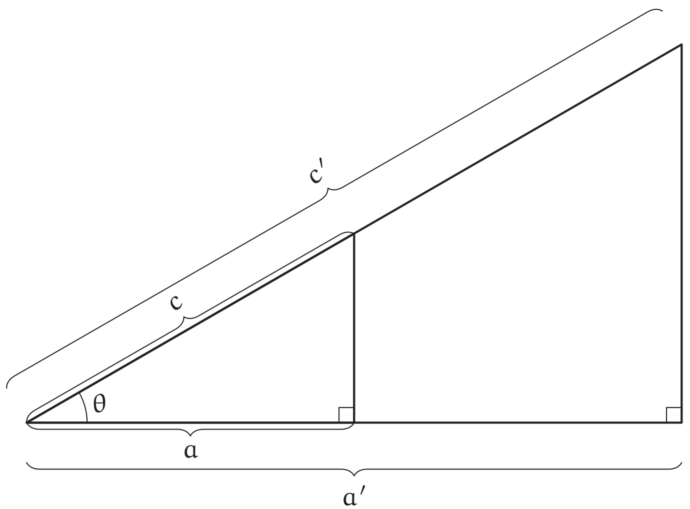
Tetra’s eyes glimmer, as if she’s come to a remarkable realization.

1.7 The Cosine Function

Me “Once you’ve come to grips with the sine function, the cosine function is a piece of cake. The only difference is that with the cosine function we’re interested in sides a and c.”



Me "We're going to stretch side c without letting $\angle\theta$ change, just like before. For example, let's double the length of side c. Then here's what happens to the triangle."



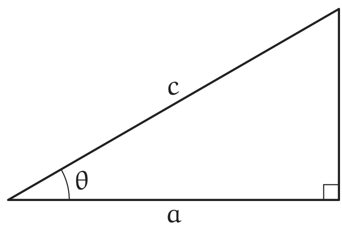
Tetra "I guess side a doubled in length, too, to a'?"

Me "Right. And this time we're saying that if the measure of $\angle\theta$ is kept constant, then the ratio of sides a and c will remain constant. The function $\cos\theta$ tells us the value of the fraction $\frac{a}{c}$ when we specify a value for θ."

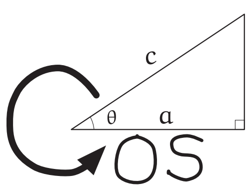
Tetra "So except for that one letter, it's just like $\sin\theta$!"

Definition of $\cos\theta$ as a ratio of two sides in a right triangle
$(0^\circ < \theta < 90^\circ)$

$$\cos\theta = \frac{a}{c}$$



Me "There's even a similar trick to remember it, using the 'c' for 'cosine.' This time, just say 'c divides a.' "



How to remember $\cos\theta$

Tetra "Without forgetting to put $\angle\theta$ on the left!"

Me "Good catch. And that's pretty much the basics of sine and cosine."

Tetra "Cool! I understand everything so far!"

1.8 Removing Conditions

Me “The next thing we want to do is get out from under this constraint that $0° < \theta < 90°$.”

Tetra “Why’s that?”

Me “Because it puts limits on what $\angle\theta$ can be, which makes it harder to deal with.”

Tetra “The fewer conditions the better, I guess. I always forget those.”

Me “You see where the condition comes from, right? Based on how we defined the sine function?”

Tetra “I do! It’s because if θ became $90°$ or larger, we wouldn’t have a right triangle any more!”

Me “That’s right. That also means that we can’t use triangles to define the sine function any more.”

Tetra “Well then what can we do?”

Me “Use circles instead.”

Tetra “But how? Circles don’t have angles, so . . . are there two kinds of sine?”

Me “Two kinds?”

Tetra “Like, one for triangles and one for circles.”

Me “No, no. They’re all the same. We’re going to work things so that when θ is in the range $0° < \theta < 90°$ nothing at all changes.”

Tetra “That sounds . . . hard.”

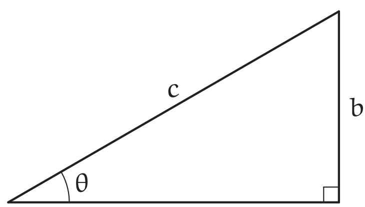
Me “Not really. We just need to make some minor adjustments.”

Tetra “If you say so . . . ”

Me “Some quick review first. Up until now, we’ve been defining the sine function as the ratio of the lengths of two sides of a triangle. A fraction, in other words.”

Definition of $\sin\theta$ as a ratio of two sides in a right triangle
$(0^\circ < \theta < 90^\circ)$

$$\sin\theta = \frac{b}{c}$$



Tetra “That looks right.”

Me “The important thing here is the value of that fraction $\frac{b}{c}$, so let’s make things easy and think of the length of side c as being exactly 1.”

Tetra “Why do we want to do that?”

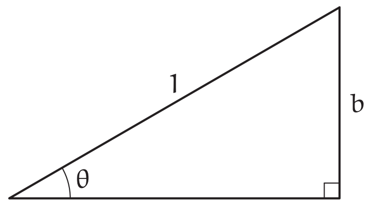
Me “Because if $c = 1$, then $\sin\theta = \frac{b}{c} = \frac{b}{1} = b$. So nice and simple.”

Tetra “Uh, okay . . . ”

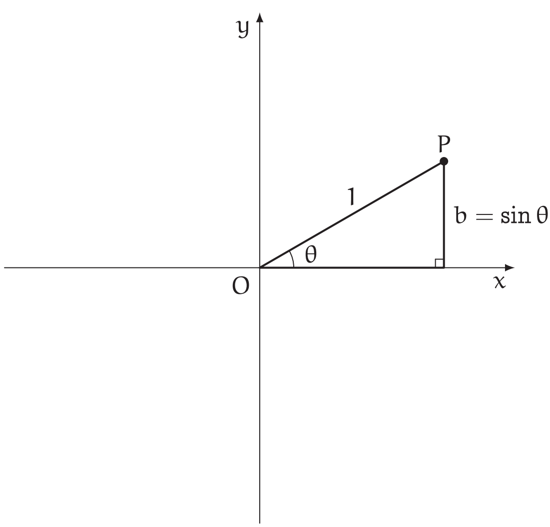
Me “This also means that since $\sin\theta = b$, the value of the sine function equals one of the side lengths.”

Definition of $\sin\theta$ as a ratio of two sides in a right triangle
$(0^\circ < \theta < 90^\circ)$

$$\sin\theta = b \qquad (\text{when } c = 1)$$



Me "Let's put our right triangle on a coordinate plane, with the vertex of $\angle\theta$ on the origin and the right angle on the x-axis. Then let's name the point that isn't on an axis P."



Placing the right triangle on a coordinate plane

Me "We've decided that $c = 1$, so the 'height' of point P equals $\sin\theta$."

Tetra "The height?"

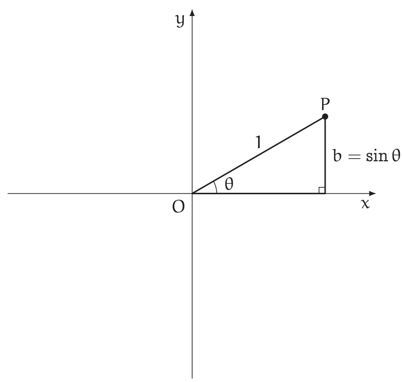
Me "The distance above the x-axis."

Tetra "Ah, okay."

Me "So here's a question. If we keep increasing $\angle\theta$, what kind of shape will point P trace out?"

A question

As we vary $\angle\theta$, what shape does point P describe?



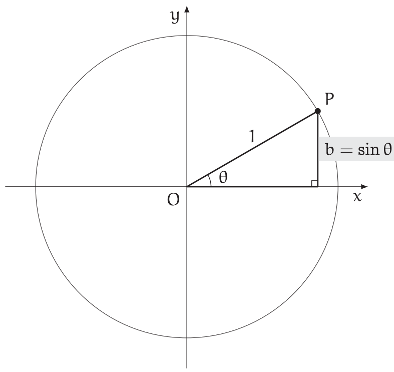
Tetra "I guess it will swing around ... in a circle?"

Me "Right! The distance between point O and point P will still be 1, so point P will move in a circle. Just like drawing one with a compass."

Tetra "Ooh, that's fun!"

Answer

As $\angle\theta$ varies, point P describes a circle.



Me “A circle with a radius of 1 is called a unit circle. We’ve centered this one on the origin.”

Tetra holds up a hand to stop me and writes in her notebook.

Tetra “Unit circle . . . Okay, I’m good. But why are we making circles?”

Me “Because they let us escape from the confines of the right triangle.”

Tetra “The right triangle was confining us?”

Me “Sure, so long as we were relying on it to define $\sin\theta$. For example, it would be nice if the sine function had a value when $\theta = 0^\circ$, but we can’t make a triangle in that case.”

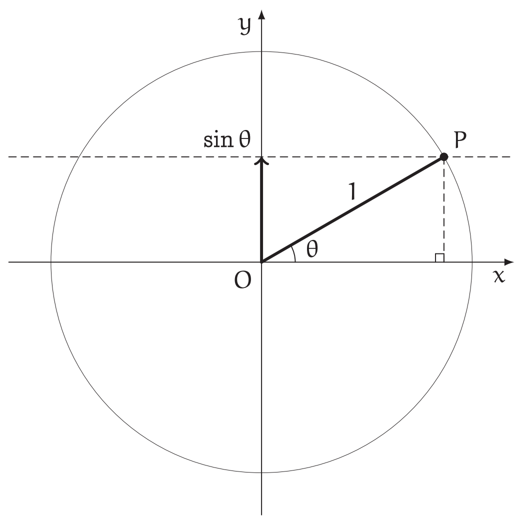
Tetra “Ah, because that would flatten the triangle out into a line.”

Me “That’s right. But if we define $\sin\theta$ using a circle, we can say its value is just the y-value of this point P.”

Tetra "Where did that come from?"

Me "Just look at the graph. You'll see."

Defining $\sin\theta$ as the y-coordinate of a point P on the unit circle.



Tetra "Hmm . . . "

Me "Do you see how when we define it like this, nothing's changed from when we used the triangle for angles $0^\circ < \theta < 90^\circ$?"

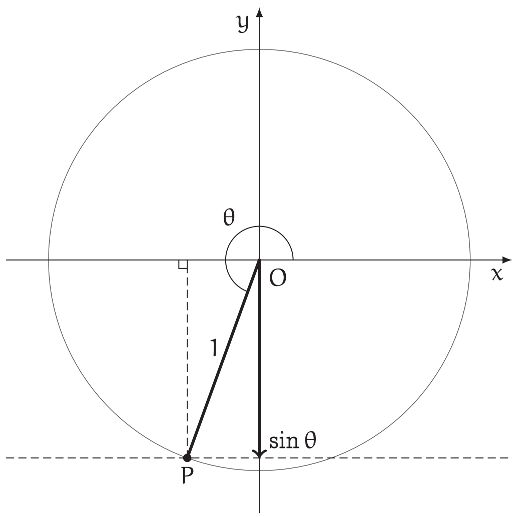
Tetra "Sure! I can still see the triangle and everything. And now I totally get what you mean by the height of that point."

Me "Maybe I shouldn't have used that word, though, since sometimes that 'height' will be negative."

Tetra "Negative height?"

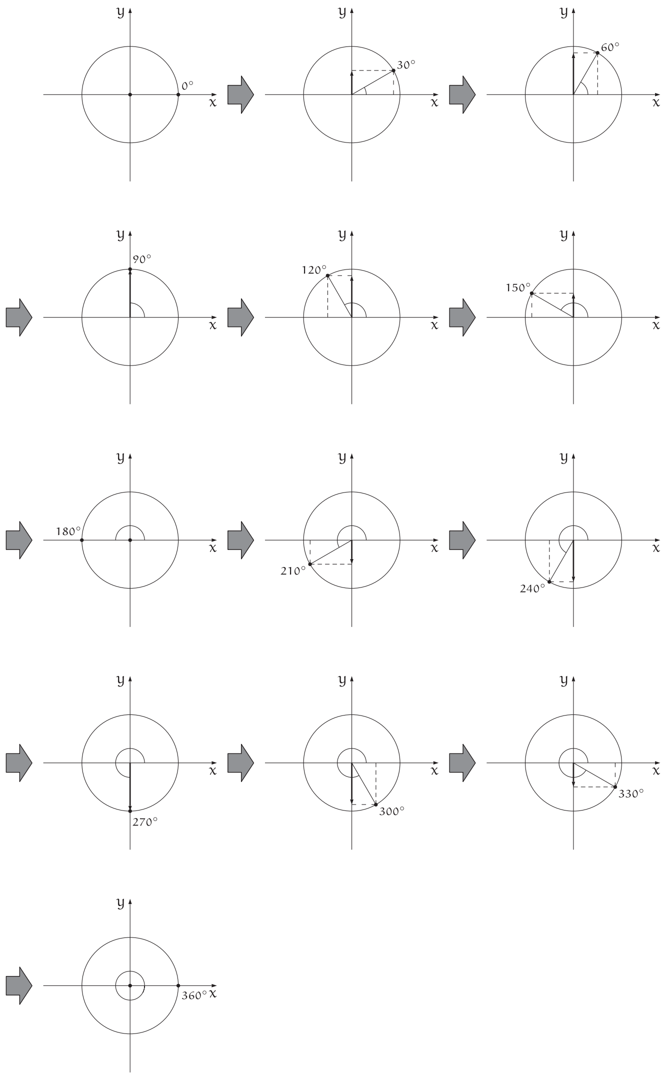
Me "It can be. Like when θ looks like this."

A case where $\sin\theta < 0$

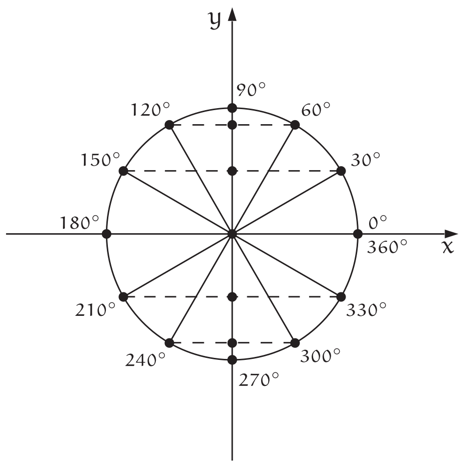


Tetra "I see! That happens when the point dips down below the x-axis."

Me "It's easy to see what's going on if you watch θ gradually increase, like this."


y
x
0°
30°
60°
90°
120°
150°
180°
210°
240°
270°
300°
330°
360°

Point locations on the unit circle as θ increases in 30° increments



Tetra "Ooh! Ooh! I think I found something! Is this true?"

$$-1 \leqslant \sin\theta \leqslant 1$$

Me "Nice! How'd you get that?"

Tetra "Well, since the point stays on a circle whose radius is 1, the y-coordinate will be 1 when it's on the top, and -1 when it's on the bottom. You said that $\sin\theta$ equals the value of the y-coordinate, so that should mean it can never get bigger than 1, or smaller than -1."

Me "A wonderful discovery, and you're absolutely right—$-1 \leqslant \sin\theta \leqslant 1$ will be true for any value of θ. That's a property that comes straight out of its definition."

1.9 Sine Curves

Miruka "You guys seem to be having fun."

Tetra "Hi, Miruka! Look! I'm learning trigonometry!"

Miruka often joined Tetra and me in our after-hours math talks. She cocks her head and peeks at what we'd been writing.

Miruka "Hmph. No sine curves yet."

Tetra "What's a sine curve?"

Miruka sits next to Tetra and snatches the pencil from my hand. She acts as cool as ever, but I can see through that—she's just itching to take the teacher's seat.

Miruka "What are the horizontal and vertical axes you're using in this coordinate plane?"

Tetra "Um . . . the x-axis and the y-axis?"

Miruka "Good. So for a point (x, y) on the unit circle, the circle represents the relationship between x and y."

Me "It's a restriction on what they can be."

Tetra "Sure, just like when we were using quadratic functions to draw parabolas."[2]

Miruka "Let's make a new graph, one where the horizontal axis is a θ-axis instead of an x-axis. We'll keep the y-axis as it is."

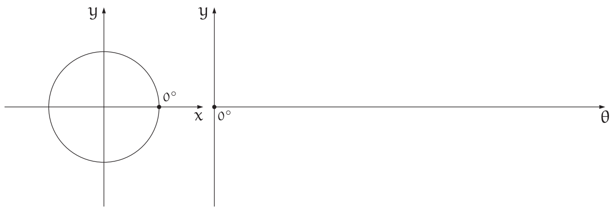
Tetra "A θ-axis?"

Me "You'll see what she means."

Miruka smiles and nods. I see that math-mode twinkle in her eye.

Miruka "Say that the angle is $0°$. Then we plot points on each graph like this."

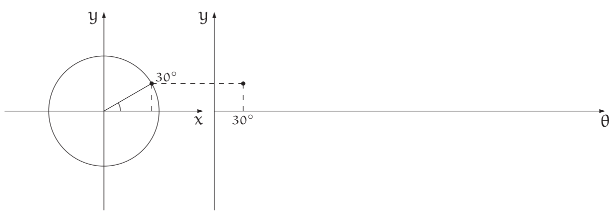
[2]See Chapter 5 of *Math Girls Talk About Equations and Graphs*



Plotting points on each graph when $\theta = 0°$

Tetra "Lemme see if I've got this straight. The graph on the right has a horizontal θ-axis, so the coordinates there are (θ, y). That means when $\theta = 0°$ we say $y = \sin 0°$, which is 0. So there's a point at $(0, 0)$. Is that right?"

Miruka "Perfect. Now let's do it with $\theta = 30°$. When we make θ bigger, the point is going to move in different ways on the two graphs. On the left graph, the point will spin around in a circle. On the right, it will keep advancing."

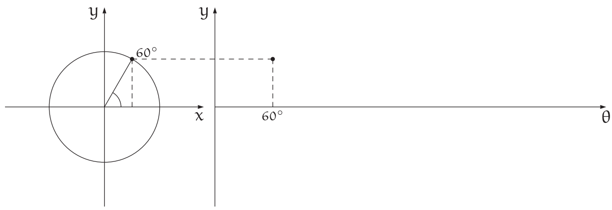


Tetra "It advances... Oh, I see that. But the height of the point will be the same on both graphs, right?"

Me "Sure, because they're both using the same y-axis."

Miruka "On to $\theta = 60°$. On the left we've spun twice as far along the circle, while on the right we've moved ahead twice as far."

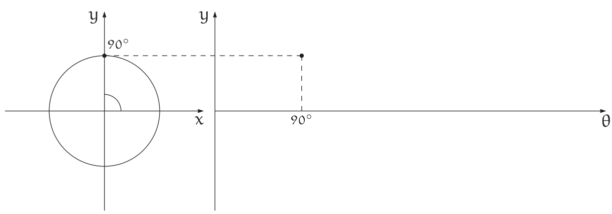
Tetra "Ah, okay, this thing about a θ-axis is starting to make more sense."



Miruka "Add another 30°, so $\theta = 90°$ now."

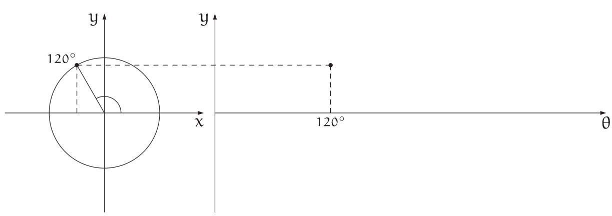
Tetra "And $\sin 90° = 1$! We've reached the top of the circle!"

Me "Or the maximum value of $\sin\theta$, if you want to look at it that way."



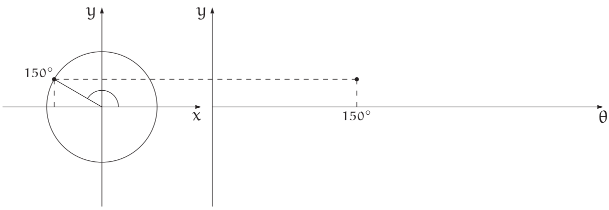
Miruka "Another 30°, to $\theta = 120°$."

Tetra "What goes up, must come down."



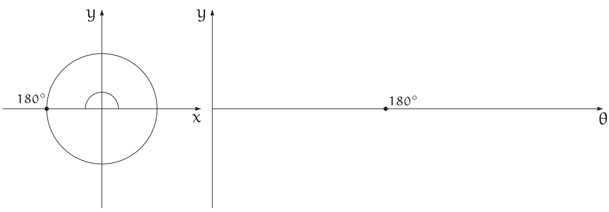
Miruka "And another 30° ..."

Tetra "Up to $\theta = 150°$ now."



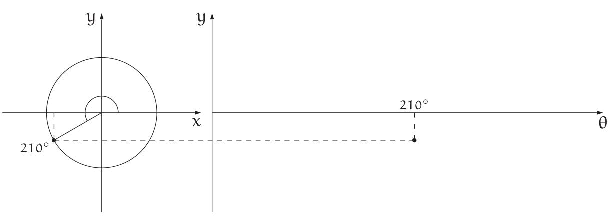
Miruka “And now at $\theta = 180^\circ \ldots$”

Tetra “Boom, we hit the floor. So $\sin 180^\circ = 0$.”

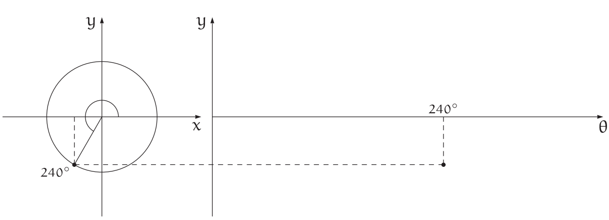


Me “We’re going to crash right through that floor, though.”

Tetra “Oh, right! Now’s when we dip under the x-axis and go negative. I guess we’re up to $\theta = 210^\circ$ next?”



Miruka “And then on to 240°.”

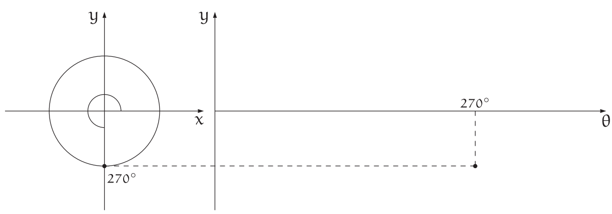


Tetra "Wow, I don't think I've ever used an angle like 240° before."

Miruka "Because of symmetry."

Tetra "Symmetry?"

Miruka "Later. Let's move ahead to 270°."

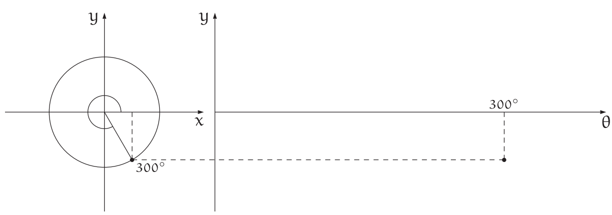


Tetra "And we've hit rock bottom. So $\sin 270^\circ = -1$!"

Me "The minimum value for $\sin\theta$."

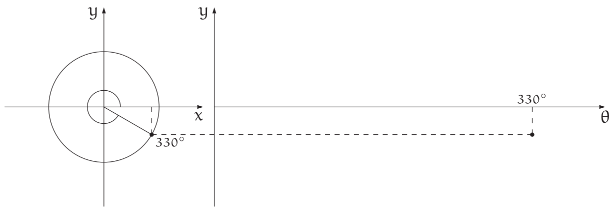
Tetra "So $\sin\theta$ is biggest when $\theta = 90^\circ$, and smallest when $\theta = 270^\circ$. Hang on, I gotta write this down ..."

Miruka "While you're doing that, I'm moving on to 300°."



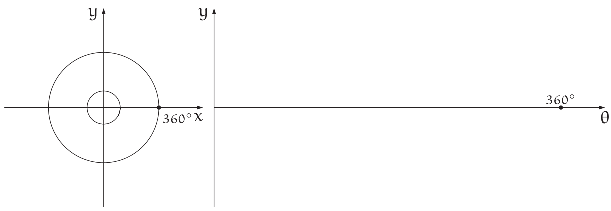
Miruka "And to 330°."

Tetra "We're repeating ourselves, aren't we? We pass through the same heights as we go up and then back down, and then do the same thing with negative heights when we're under the axis."



Miruka “We end up back at home with 360°.”

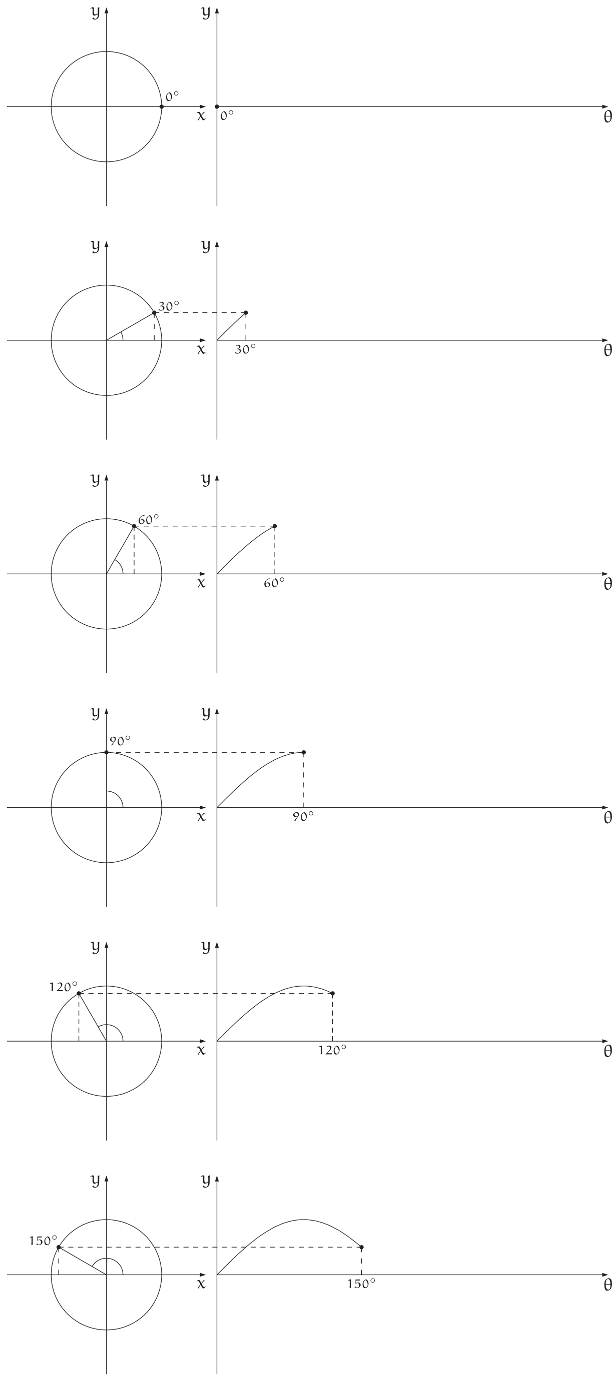
Tetra “All the way around and back where we started. So $\sin 360^\circ = 0$.”



Me “The big question is, do you see the sine curve?”

Tetra “I do! It comes from the up-down-up on the left making waves on the right!”

Miruka “And that wave is the sine curve. If we draw the curve instead of just plotting points, here’s what we get.”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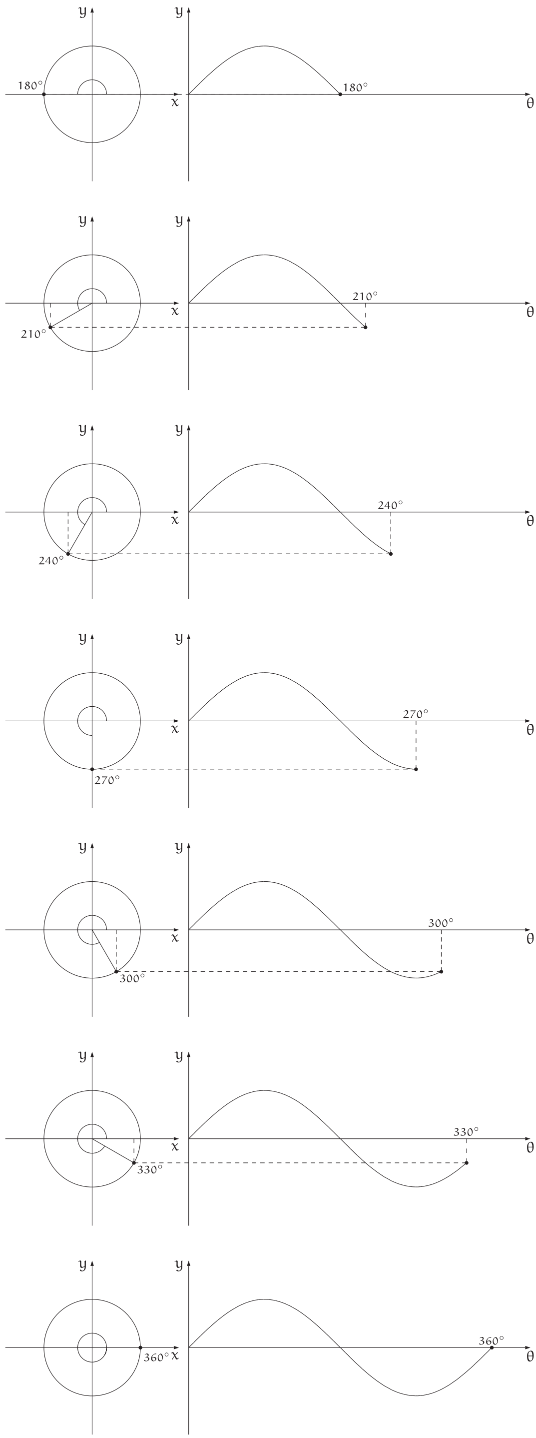
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y
300°
x
θ
300°
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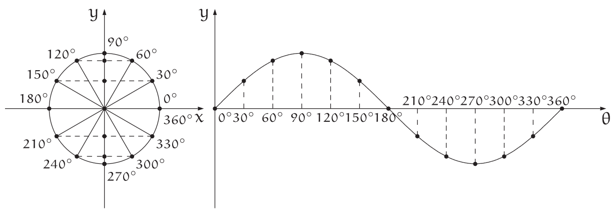
y
330°
x
θ
330°
y
y
360°
360°
x
θ

Tetra "It's so . . . pretty!"

Me "I agree. It is pretty."

Miruka "It's *beautiful*. Such an elegant correspondence between the unit circle and the sine curve."

Correspondence between the unit circle and the sine curve



Tetra "So what does the cosine curve look like?"

Miruka "The cosine curve?"

Tetra "Well, we just graphed $\sin\theta$ to get the sine curve, right? Seems like we should be able to graph $\cos\theta$ and get a cosine curve, too."

Miruka "You can graph $\cos\theta$, but you don't call that a cosine curve. It gives you another sine curve."

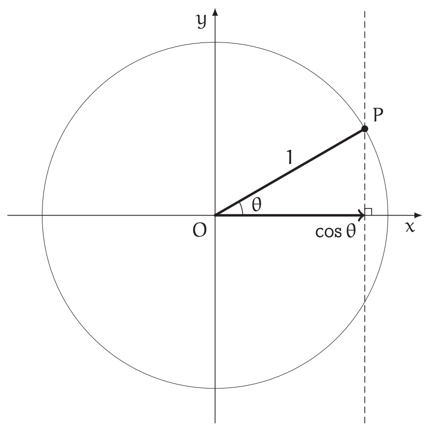
Tetra "Huh? Why?"

Miruka "The graphs of $\sin\theta$ and $\cos\theta$ are similar, with just one minor difference. I'm pretty sure you could graph it yourself if you tried."

Tetra "You sure about that?"

Miruka "When you defined $\sin\theta$ using a circle, the value of $\sin\theta$ was just the y-coordinate of a point on the circle, right? Well $\cos\theta$ is the x-coordinate. Knowing that should be enough to graph the cosine function."

Defining $\cos\theta$ as the x-coordinate of a point P on the unit circle.



Ms. Mizutani "The library is *closed*!"

With her announcement that it was time to go home, Ms. Mizutani shut down yet another math talk. As I walked home, I wondered if Tetra would be successful in graphing the cosine function . . .

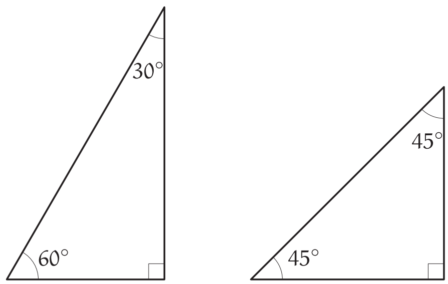
"If name is sufficient to represent form, then a name is all you need."

Appendix: Greek Letters

Lowercase	**Uppercase**	**Name**
α	A	alpha
β	B	beta
γ	Γ	gamma
δ	Δ	delta
$\epsilon \quad \varepsilon$	E	epsilon
ζ	Z	zeta
η	H	eta
$\theta \quad \vartheta$	Θ	theta
ι	I	iota
$\kappa \quad \varkappa$	K	kappa
λ	Λ	lambda
μ	M	mu
ν	N	nu
ξ	Ξ	xi
o	O	omicron
$\pi \quad \varpi$	Π	pi
ρ	P	rho
σ	Σ	sigma
τ	T	tau
υ	Υ	upsilon
$\phi \quad \varphi$	Φ	phi
χ	X	chi
ψ	Ψ	psi
ω	Ω	omega

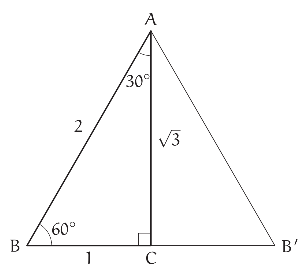
Appendix: Values of trigonometric functions for common angles

The angles 30°, 45°, and 60° are frequently used in a variety of problems. Let's find the values of the sine and cosine functions for those angles.



Right triangles with 30°, 45°, and 60° angles

We'll start with the 30° and 60° angles. If you place two right triangles having a 60° angle back-to-back, you get a triangle $\triangle ABB'$ with three 60° angles, as in the diagram below. Since each angle has the same measure $\triangle ABB'$ is an equilateral triangle, so we know that $\overline{BB'} = \overline{AB}$.



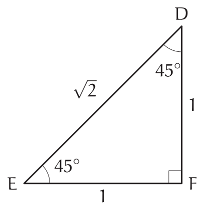
Letting $\overline{BC} = 1$, we have that $\overline{B'C} = 1$ and $\overline{BB'} = \overline{AB} = 2$. Let's apply the Pythagorean theorem to the right triangle $\triangle ABC$ to find $\overline{AC}$.

$$\begin{aligned} \overline{BC}^2 + \overline{AC}^2 &= \overline{AB}^2 && \text{the Pythagorean theorem} \\ 1^2 + \overline{AC}^2 &= 2^2 && \text{because } \overline{BC} = 1 \text{ and } \overline{AB} = 2 \\ \overline{AC}^2 &= 3 \\ \overline{AC} &= \sqrt{3} \end{aligned}$$

From this we find that $\overline{AC} = \sqrt{3}$. Now we can find the other values:

$$\begin{aligned} \cos 30^\circ &= \frac{\overline{AC}}{\overline{AB}} = \frac{\sqrt{3}}{2} \\ \cos 60^\circ &= \frac{\overline{BC}}{\overline{AB}} = \frac{1}{2} \\ \sin 30^\circ &= \frac{\overline{BC}}{\overline{AB}} = \frac{1}{2} \\ \sin 60^\circ &= \frac{\overline{AC}}{\overline{AB}} = \frac{\sqrt{3}}{2} \end{aligned}$$

Now let's look at 45° angles. In $\triangle DEF$, the measure of both $\angle D$ and $\angle E$ is 45°, so $\triangle DEF$ is an isosceles triangle with $\overline{DF} = \overline{EF}$.



Letting $\overline{DF} = \overline{EF} = 1$, we can use the Pythagorean theorem again to find $\overline{DE}$:

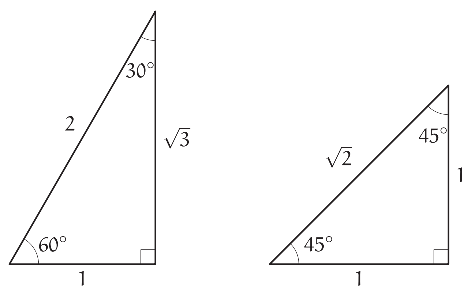
$$\begin{aligned} \overline{DF}^2 + \overline{EF}^2 &= \overline{DE}^2 && \text{the Pythagorean theorem} \\ 1^2 + 1^2 &= \overline{DE}^2 && \text{because } \overline{DF} = \overline{EF} = 1 \\ \overline{DE}^2 &= 2 \\ \overline{DE} &= \sqrt{2} \end{aligned}$$

So we've found that $\overline{DE} = \sqrt{2}$. From this, we can find the following values:

$$\cos 45^\circ = \frac{\overline{EF}}{\overline{DE}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$\sin 45^\circ = \frac{\overline{DF}}{\overline{DE}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

These results are summarized below:

Values of trigonometric functions for some common angles



θ	30°	45°	60°
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$

Problems for Chapter 1

"First, we have to *understand* the problem; we have to see clearly what is required."

George Pólya
How To Solve It, 2nd Ed.

Problem 1-1 (Finding $\sin\theta$)

Find the value of $\sin 45^\circ$, without looking at the appendix to this chapter.

(Answer on page 227)

Problem 1-2 (Finding θ from $\sin\theta$)

Find all possible values for θ in the range $0^\circ \leqslant \theta \leqslant 360^\circ$ for which $\sin\theta = \frac{1}{2}$.

(Answer on page 228)

Problem 1-3 (Finding $\cos\theta$)

Find the value of $\cos 0^\circ$.

(Answer on page 230)

Problem 1-4 (Finding θ from $\cos\theta$)

Find all possible values for θ in the range $0^\circ \leqslant \theta \leqslant 360^\circ$ for which $\cos\theta = \frac{1}{2}$.

(Answer on page 230)

Problem 1-5 (Graphing $x = \cos\theta$)

Draw a graph of $x = \cos\theta$ for values of θ in the range $0^\circ \leqslant \theta \leqslant 360^\circ$. In the graph, use values of θ for the horizontal axis, and values of x for the vertical axis.

(Answer on page 232)

About this sample chapter

Thank you for reading this sample chapter from *Math Girls Talk About Trigonometry*. I hope you enjoyed reading it as much as I enjoyed translating it.

This is the third volume in a series of books that aims at addressing select topics from a variety of areas of mathematics. Alongside some fundamental concepts, these books present fun excursions that learners are less likely to see in typical classroom settings. They attempt to illustrate problem solving techniques in those areas, and to show how fun and exciting mathematics can be (especially when enjoyed with friends!). The presentation is via dialogue between characters from the *Math Girls series* of books, which generally contain higher level math.

The first volume in this series, *Math Girls Talk About Equations & Graphs*, develops topics such as using variables in equations, polynomials, setting up systems of equations, proportions and inverse proportions, the relation between equations and their graphs, parabolas, intersections, and tangent lines. The second volume, *Math Girls Talk About Integers*, covers topics from discrete mathematics such as finding primes, the interesting mathematics behind a magic trick, and using remainders from division to solve a problem based on clocks. A "final boss" chapter tackles a challenging proof by mathematical deduction, using a problem from an actual Japanese college entrance examination. Besides what you've just read, other chapters in this book include fun explorations of Lissajous curves, methods of estimating pi, and using vectors to find the trigonometric addition formulas.

More books are planned, covering topics such as sequences and series, the mathematics of change, math with vectors, and probability. We hope you enjoy them all.

If you would like to purchase the full version of this book, you can order it directly from our printer, via major online booksellers, or by special order at your local book store. We hope you enjoy it, and the rest of the *Math Girls* books!

Tony Gonzalez

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Index

Other works by Hiroshi Yuki

(in English)

- *Math Girls*, Bento Books, 2011
- *Math Girls 2: Fermat's Last Theorem*, Bento Books, 2012
- *Math Girls Manga*, Bento Books, 2013
- *Math Girls Talk About Equations & Graphs*, Bento Books, 2014
- *Math Girls Talk About the Integers*, Bento Books, 2014

(in Japanese)

- *The Essence of C Programming*, Softbank, 1993 (revised 1996)
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- *An Introduction to CGI with Perl, Basics*, Softbank Publishing, 1998
- *An Introduction to CGI with Perl, Applications*, Softbank Publishing, 1998

- *Java Programming Lessons (Vols. I & II)*, Softbank Publishing, 1999 (revised 2003)
- *Perl Programming Lessons, Basics*, Softbank Publishing, 2001
- *Learning Design Patterns with Java*, Softbank Publishing, 2001 (revised and expanded, 2004)
- *Learning Design Patterns with Java, Multithreading Edition*, Softbank Publishing, 2002
- *Hiroshi Yuki's Perl Quizzes*, Softbank Publishing, 2002
- *Introduction to Cryptography Technology*, Softbank Publishing, 2003
- *Hiroshi Yuki's Introduction to Wikis*, Impress, 2004
- *Math for Programmers*, Softbank Publishing, 2005
- *Java Programming Lessons, Revised and Expanded (Vols. I & II)*, Softbank Creative, 2005
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- *Revised Perl Programming Lessons, Basics*, Softbank Creative, 2006
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- *Math Girls Secret Notebook / Let's Play with the Integers*, Softbank Creative, 2013
- *The Birth of Math Girls,* Softbank Creative, 2013
- *Math Girls Secret Notebook / Round Trigonometric Functions*, Softbank Creative, 2014