MATH GIRLS

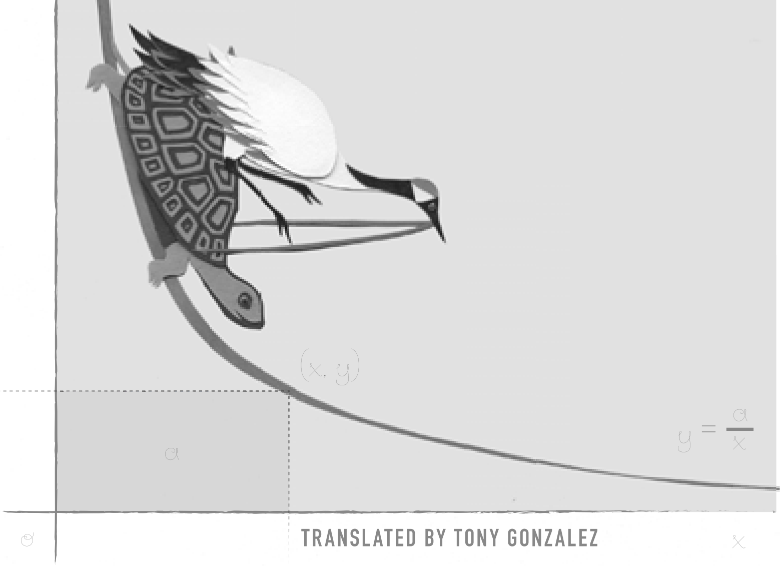
TALK ABOUT

EQUATIONS & GRAPHS

Fundamental Skills for Advanced Mathematics

BY HIROSHI YUKI

Author of MATH GIRLS



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Math Girls Talk About
Equations and Graphs

To My Readers

This book is a collection of conversations between Miruka, Tetra, Yuri, and our narrator.

If there are places where you don't understand what they're talking about, or equations you don't understand, feel free to skip over those parts. But please do your best to keep up with them.

That's the best way to make yourself part of the conversation.

—Hiroshi Yuki

Cast of Characters

I am your narrator. I'm a junior in high school, and I love math. Equations in particular.

Miruka is my age. She's so good at math, it's scary. She has long black hair and wears metal frame glasses.

Tetra is one year younger than me, and a bundle of energy. She cuts her hair short and has big, beautiful eyes.

Yuri is my cousin, an eighth grader. She has a chestnut ponytail and excels at logic.

Ms. Mizutani is our school librarian.

Mom is, well, just my mom.

Contents

Prologue

Conversations contain everything. Doubts and answers, agreement and arguments, praise and criticism. Space. Time. And secrets.

We share our secrets through conversation. We share the secrets hidden in equations and graphs. Secrets hidden in identities and simultaneous equations. In parabolas and hyperbolic curves.

Math isn't about whether you can solve a given problem. It's about tackling things head on and thinking deeply. It's about asking questions and finding answers.

The conversations we have together become our new secrets, replacing the secrets they helped us uncover. Precious secrets that no one else will ever know, and no one can ever steal.

These are our conversations.

CHAPTER 1

Letters and Identities

"Time to become a problem solver."

1.1 Bugged by Letters

Tetra "There you are!"

Me "Hey, Tetra. Looks like something's got you psyched."

Tetra "You know me."

Me "But this *is* the library. Maybe we should keep it down a bit?"

Tetra "Oh, right."

Tetra was by far the giddiest of my friends. She was in tenth grade, one year behind me. She cut her hair short and had big, beautiful eyes. We met in the library after school almost every day, where we talked about all kinds of things. Well, all kinds of mathy things, at least.

Tetra "Okay to ask you about something?"

Me "Of course. What is it?"

Tetra sits down next to me.

Tetra "When I study math, I always get confused about when to use what letters. I look at the equations in my book and wonder, 'Why did they use these letters? Why didn't they use something else?'"

Me "Give me an example."

Tetra "An example?"

Me "Sure. Show me an equation where the letters bug you."

Tetra "Hard to do off the top of my head."

Me "Well then just give me any equation, and let's talk about the letters in it."

Tetra "Okay, let's see..."

Tetra looks up while she thinks.

Tetra "How about this one?"

$$(a+b)(a-b) = a^2 - b^2$$

Tetra "That's how my teacher wrote it on the board. But in my book, it looks like this."

$$(x+y)(x-y) = x^2 - y^2$$

Me "That's an example of an identity."

Tetra "A whatity?"

Me "An identity. An equation that will always be true, no matter what numbers you stick into it. This equation $(a+b)(a-b) = a^2 - b^2$ always holds, no matter what a and b are, so it's an identity. If you want to be precise, it's an identity on a and b."

Tetra "Gotcha!"

Me "This is a good one to remember, by the way. Just think, 'the product of a sum and a difference is a difference of squares.' So when you have a sum $a+b$ and a difference $a-b$, if you multiply those two together you'll get a^2-b^2. Since that's an identity, it's always true, no matter what values a and b have."

Tetra "Well what about $(x+y)(x-y)=x^2-y^2$?"

Me "Same thing. This one is an identity on x and y, so that's a true statement, no matter what x and y are."

Tetra "So which one is right, a and b, or x and y? When do I use one or the other?"

Me "In this case, it doesn't really matter."

$$(a+b)(a-b)=a^2-b^2 \quad \text{an identity on } a \text{ and } b$$
$$(x+y)(x-y)=x^2-y^2 \quad \text{an identity on } x \text{ and } y$$

Tetra "So I've been worrying all this time about something that doesn't matter?"

Tetra crosses her arms.

Me "This thing about the letters really *does* bug you, doesn't it."

Tetra "Yeah. I think my problems with math began when these letters started popping up. Now it's a, b, c this and x, y, z that. I even see Greek letters sometimes!"

Me "I hadn't really thought about it, but you're right. After a point, math is more about letters than numbers."

Tetra "Maybe I'm just being silly. I'm slow enough as it is without worrying about this kind of thing."

Me "You aren't slow, you're careful. There's a big difference. But the important thing isn't which letters are being *used*, it's what they *mean*. That's what you really need to pay attention to."

Tetra “What do you mean, what they mean?”

Me “Math is something you have to read carefully, not skim through. You need to go slow, and pay attention to what those letters are doing, and why they’re there.”

1.2 Where Do the Letters Reappear?

Me “Another thing to keep an eye out for is where letters reappear.”

Tetra “I never knew they *dis*appeared.”

Me “That’s not what I mean. Here, check out this identity we were talking about.”

$$(a+b)(a-b) = a^2 - b^2$$

Me “See how the a and b show up in several places? One of the rules of math is that when a letter is repeated, it has to mean the same thing. So the a in the $(a+b)(a-b)$ on the left side of this equation and the a in the $a^2 - b^2$ on the right have to represent the same number.”

Tetra “Wait, you lost me. You’re saying that a is always the same number?”

Me “Not quite. I’m saying it’s always the same number in *this* equation. As long as we’re talking about the identity $(a+b)(a-b) = a^2 - b^2$, every a has to represent the same number, and so does every b. Of course a and b might represent the same number too.”

Tetra “Okay, I see what you’re saying, but—”

Me “But why am I reviewing something you learned years ago?”

Tetra “Something like that, yeah.”

Me “Because I want to show you how something cool happens when you plug numbers into those a’s and b’s. Let’s start with $a = 100, b = 2$. That means we’re substituting a 100 for all the a’s and a 2 for all the b’s.”

Tetra “How do you know those are right answers?”

Me “Because this is an identity, remember? That means it works for *any* numbers we pick.”

Tetra “Oh, of course.”

Me “Here’s what we get after substituting.”

$$
\begin{aligned}
(a+b)(a-b) &= a^2 - b^2 && \text{“product of a sum and a difference”}\\
(\underline{100}+b)(\underline{100}-b) &= \underline{100}^2 - b^2 && \text{substitute } a\text{’s with 100 as an example}\\
(100+\underline{2})(100-\underline{2}) &= 100^2 - \underline{2}^2 && \text{substitute } b\text{’s with 2 as an example}\\
\underline{102} \times \underline{98} &= 100^2 - 2^2 && \text{calculate the left side}\\
102 \times 98 &= \underline{10000} - \underline{4} && \text{calculate the right side}
\end{aligned}
$$

Tetra “Okay, so?”

Me “So we found this equation.”

$$102 \times 98 = 10000 - 4$$

Tetra “Sorry, I still don’t see where this is headed.”

Tetra gives a doubtful frown.

1.3 Mental Arithmetic

Me “Let’s start with the left side of that equation.”

$$\underbrace{102 \times 98}_{\text{left side}} = 10000 - 4$$

Me “Calculating 102×98 in your head is kinda tough, right?”

Tetra “Yeah, wow. Let’s see, 8×2 is—”

Me “No, you don’t have to do all that. Look at the right side of the equation.”

$$102 \times 98 = \underbrace{10000 - 4}_{\text{right side}}$$

Me “It’s just $10000 - 4$. Much easier.”

Tetra “Sure, 9996, right?”

Me “See how we’ve turned something hard like 102×98 into a simple subtraction problem?”

Tetra “I think so.”

Me “The trick is looking at 102×98, and noticing that 102 is $100 + 2$, and that 98 is $100 - 2$. Once you start playing around and plugging actual numbers into an identity like $(a + b)(a - b) = a^2 - b^2$, you’ll get the hang of it in no time. Half the things they teach you in those mental arithmetic books rely on identities like this. Kinda fun, really.”

Miruka “What’s kinda fun?”

Miruka appears out of nowhere, and I nearly fall out of my seat.

Tetra “Miruka, stop sneaking up on us like that!”

Me “At least make some sound when you walk.”

Miruka “Yeah, yeah. So tell me, what’s fun?”

Miruka was a classmate. She had long, black hair and wore metal frame glasses. When it came to math, she had us all beat—but it was more than that. Miruka had this certain poise that made it hard for me to tear my eyes away from her.

Miruka peers at the notebook I've been writing in.

Miruka "'The product of a sum and a difference is a difference of squares.' Hmph."

Tetra "We were studying identities."

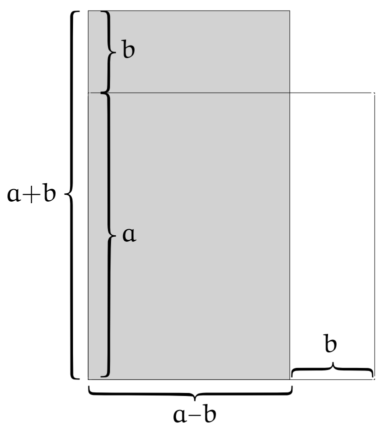
$$(a+b)(a-b) = a^2 - b^2$$

Miruka "About how this identity changes rectangles into squares?"

Tetra "Uh, not exactly."

Miruka "Here, take a look."

Miruka sketches a quick graph.



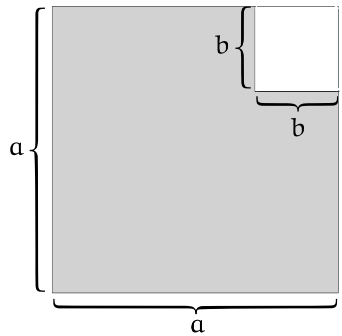
$(a+b)(a-b)$ as a rectangle

Me "Oh, I get it!"

Tetra "I don't. What's this a graph of?"

Me "A rectangle where the length of one side is $a + b$, and the other is $a - b$. So its area is the product of the two, $(a + b)(a - b)$! Of course!"

Miruka "And this one is a figure for $a^2 - b^2$. You can make it using the two squares."



$a^2 - b^2$ as two squares

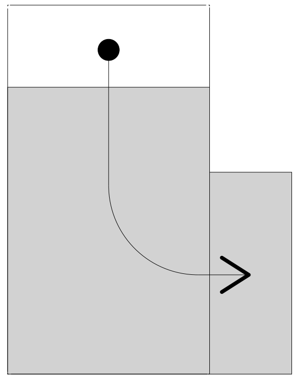
Tetra "Two squares? Where did they come from?"

Me "You start with a big square with area a^2, then chop out a square corner that has an area of b^2."

Miruka "Precisely."

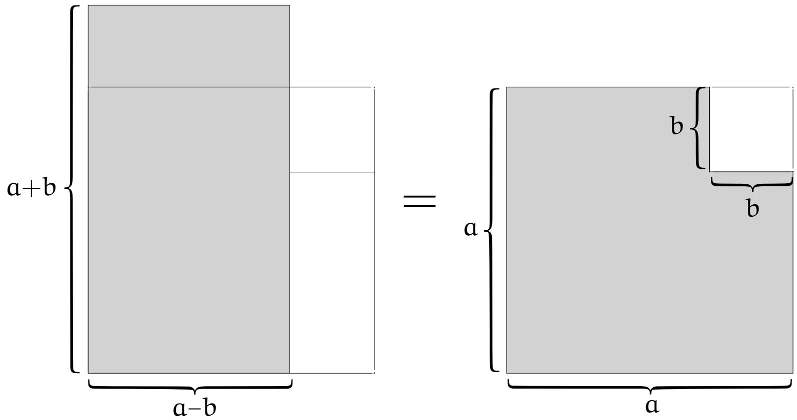
Miruka's eyes gleam.

Tetra "Oh! You just moved the smaller rectangle down and to the right. And just moving it doesn't change the area!"



Me "This shows you exactly what's going on in the equation $(a+b)(a-b) = a^2 - b^2$. The $(a+b)(a-b)$ on the left is the area of the big rectangle. The $a^2 - b^2$ on the right is the area of the big square with the area of the little square taken out."

Tetra "Makes perfect sense!"



A graphical representation of $(a+b)(a-b) = a^2 - b^2$

Me — "This also explains why $(a+b)(a-b) = a^2 - b^2$ is an identity."

Miruka cocks her head.

Miruka — "Well, don't get too carried away. This geometric explanation only works if a and b are real numbers, and greater than zero."

Tetra — "So you can't do this all the time?"

Miruka — "Yes and no. The identity $(a+b)(a-b) = a^2 - b^2$ is supposed to be true for *any* values of a and b. But there are some assumptions built into this graph, namely that $a \geqslant 0$ and $b \geqslant 0$. Do you see why?"

Tetra — "Because we're talking about side lengths, and those can't be negative?"

Me — "That's right. Actually there's another assumption in this graph, that $a \geqslant b$. A picture's worth a thousand words, but they can make it easy to overlook the conditions that come with them. Score one for equations."

Tetra — "Is this what you meant when you were talking about the meaning behind the letters?"

Me — "Exactly. Good eye!"

Miruka — "What's all this about?"

Me — "Just before you snuck up on us, I was telling Tetra how it's important to be sure you understand what the letters in equations stand for."

Miruka — "So that's the kind of thing you two talk about."

Tetra — "He was also showing me how to stick numbers into $(a+b)(a-b) = a^2-b^2$ to do tricky calculations like 102×98.

$$
\begin{aligned}
(a+b)(a-b) &= a^2 - b^2 && \text{“product of a sum and a difference”} \\
(100+2)(100-2) &= 100^2 - 2^2 && \text{substitute } a = 100, b = 2 \\
102 \times 98 &= 10000 - 4 && \text{calculate } 102 \times 98 \text{ as } 10000 - 4
\end{aligned}
$$

Tetra “Now I can do 102 × 98 in my head!”

1.4 Expanding Equations

Me “Don't worry, Tetra. I think a lot of people get lost when letters start creeping into math. The way they teach us doesn't help.”

Tetra “It is a little confusing. In class we move stuff around to solve for x or whatever, but I have no idea what's going on. It drives me nuts.”

Me “Your teacher's just getting you to practice before you start on cooler stuff.”

Miruka “I thought you loved messing with equations.”

Me “I do. Especially expanding things out and factorizations. I liked it so much in junior high I spent hours in the library after school doing more of it. After a while, it all starts to come naturally. Good old $(a+b)(a-b)$ and I spent many an afternoon chasing after $a^2 - b^2$.”

$$
\begin{aligned}
&(a+b)(a-b) \\
&= \underline{(a+b)}a - \underline{(a+b)}b && \text{distribute the } (a+b) \\
&= \underline{aa + ba} - (a+b)b && \text{expand the } (a+b)a \text{ part} \\
&= aa + ba - \underline{(ab + bb)} && \text{expand the } (a+b)b \text{ part} \\
&= aa + ba \underline{- ab - bb} && \text{get rid of the parentheses} \\
&= aa + \underline{ab} - ab - bb && \text{rewrite } ba \text{ as } ab \\
&= aa - bb && ab - ab \text{ is 0, so it goes away} \\
&= \underline{a^2} - bb && \text{rewrite } aa \text{ as } a^2 \\
&= a^2 - \underline{b^2} && \text{rewrite } bb \text{ as } b^2
\end{aligned}
$$

Me “That’s the equations-only way of getting from $(a + b)(a - b)$ to $a^2 - b^2$. We’re just moving things around, so this should work for any values a and b. No worries about hidden conditions like with figures, either. Work that out yourself a few times, and you’ll never forget the identity.”

Tetra “Erm, if you say so.”

Me “Something wrong?”

Tetra “No, nothing. I guess I get how playing with identities can be fun. It’s all about learning new tricks for calculating stuff, right? And looking at things using graphs like Miruka did is kinda neat.”

Me “I think so, at least.”

Tetra “I guess I just have to try harder to—”

Ms. Mizutani “The library is *closed*!”

Ms. Mizutani was our school librarian. She always wore a drum-tight skirt and glasses so dark they looked like sunglasses. Every day when it was time to lock up, she marched to the center of the room and announced that the library was closed. This wasn’t the first time her proclamation had cut short one of our math talks.

“Skillful problem solvers must also be skillful readers.”

Problems for Chapter 1

Problem 1-1 (Expanding equations)

Expand the following:

$$(x + y)^2$$

(Answer on page 131)

Problem 1-2 (Calculating equations)

Letting x be 3 and y be -2, calculate the following:

$$x^2 + 2xy + y^2$$

(Answer on page 132)

Problem 1-3 (Products of sums and differences)

Calculate the following:

$$202 \times 198$$

(Answer on page 133)

Index

P

R

S

T

U

V

W

Other works by Hiroshi Yuki

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