Math Girls³

Gödel's Incompleteness Theorems

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MATH GIRLS 3: GÖDEL'S INCOMPLETENESS THEOREMS
by Hiroshi Yuki

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Gödel's Incompleteness Theorems

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Chapter 4

Know Your Limits

> "Now Cinderella, depart; but remember, if you stay one instant after midnight, your carriage will become a pumpkin, your coachman a rat, your horses mice, and your footmen lizards; while you yourself will be the little cinder-wench you were an hour ago."
>
> Charles Perrault
> *Cinderella*, trans. Unknown

4.1 At Home

4.1.1 Yuri

"Arrrgh!" Yuri bellowed, storming into my room one Saturday in February. She threw her bag across the room, where it collided with my bookshelf.

"Whoa, what's up with you?" I asked.

I'd heard Yuri enter my house and say hi to my mom, chipper as usual, so this display was more than a little unexpected.

"I let a boy get the best of me yesterday. Gah, I'm so mad. I *hate* losing."

Yuri shook her head, whipping her ponytail from side to side.

"You got in a fight?"

"No way I'd lose a fight with that dweeb. He beat me in *math*—that's what's so infuriating."

Yuri pulled a notebook out of her bag, turned to a page, and slammed it on my desk.

"This problem," she said.

Problem 5-1

Is the following a true statement?

$$0.999\cdots = 1$$

"Ah, a classic," I said. "What was your answer?"

"Something along the lines of 'of course not, moron.'"

"Why'd you say that?"

"Because it *isn't*! I mean, look at it! It's zero-point-lotsa-nines. It's gotta be a little bit less than 1!"

"And what did he say?"

"He got all full of himself, said he could prove it was true."

4.1.2 The Boy's Proof

Yuri turned the page in her notebook to reveal a proof in a handwriting that was not her own.

Answer 5-1

Clearly, 1 equals 1.

$$1 = 1$$

Divide both sides of this equation by 3, writing the left side in decimal and the right side as a fraction.

$$0.333\cdots = \frac{1}{3}$$

Multiply both sides by 3.

$$3 \times 0.333\cdots = 3 \times \frac{1}{3}$$

Calculate both sides.

$$0.999\cdots = 1$$

Thus, $0.999\cdots = 1$.

"Not bad for a middle school student," I said. "The kid has potential."

Yuri jabbed my shoulder.

"You aren't allowed to take his side! Is this right, by the way?"

"The rigor could be improved, but yeah, pretty much."

"Bah!" Yuri slumped into the chair next to my desk. "To be honest, after I got home that day I came up with my own proof. But I was hoping it'd be wrong."

"Why's that?"

"Because I want the equals sign to mean things are absolutely, positively, right-on-the-nose the same! That's what's cool about math, that it can be *right*, no questions asked. I want to say that $0.999\cdots < 1$, none of this 'pretty much equals' garbage."

"How about you show me your proof, and *then* we can talk about how all this is right or wrong."

"I've got a better idea. How about I show you my proof, and you explain how it can't be true?"

"How about we just follow the math, and see where it takes us?"

"Deal!"

4.1.3 Yuri's Proof

"Okay," I said. "Show me your proof."

"My *wrong* proof. You've just gotta show me how."

"Proceed."

"Okay, so I started thinking about how you can start with 0.9, then add a nine to get 0.99, then 0.999 and so on."

"Good so far."

"Like, 0.9 is pretty close to 1, but it's still 0.1 from getting there."

"You're talking about the difference between 1 and 0.9, yeah?"

I wrote an equation in the notebook.

$$1 - 0.9 = 0.1$$

"The difference. Exactly. Yeah, I should have used equations I guess. Here, gimme that pencil."

I handed the pencil to Yuri, and she spun the notebook toward herself.

"So there's 0.9, but then we can do 0.99."

$$1 - 0.99 = 0.01$$

"And so on and so on."

$$\begin{aligned} 1 - 0.9 &= 0.1 \\ 1 - 0.99 &= 0.01 \\ 1 - 0.999 &= 0.001 \\ 1 - 0.9999 &= 0.0001 \\ 1 - 0.99999 &= 0.00001 \\ &\vdots \end{aligned}$$

"If you repeat this *infinitely many times*, you end up with the difference with 1 being $0.000\cdots$."

$$1 - 0.999\cdots = 0.000\cdots$$

"On the right, you've got $0.000\cdots$, which is just 0, right?"

$$1 - 0.999\cdots = 0$$

"So if the difference between 1 and $0.999\cdots$ is 0, then $0.999\cdots$ must equal 1!"

$$0.999\cdots = 1$$

"And that's my proof," Yuri said, putting the pencil down.

"That's well thought out," I said. "An excellent job for someone in middle school."

"Thank you, but you really need to cut out that 'for someone in middle school' stuff. It's annoying."

"Sorry. But anyway there's one thing we definitely have to clean up—the part where you talk about doing something 'infinitely many times.' That's kind of mathematically sloppy."

"I figured. Seems like no matter how far you go, that $0.000\cdots$ is just a *liiitle* bit bigger than 0. But that would mean that $0.999\cdots$ is just a *liiitle* bit smaller than 1, so I'm kinda hoping that's the case."

"Hmm... I don't know."

"At least give me a shot at showing you how that might work."

"By all means."

4.1.4 Yuri's Alternate Proof

Yuri turned back to the notebook.

"Okay, so check this out," she said. "Obviously 0.9 is smaller than 1, right?"

$$0.9 < 1$$

"But so is 0.99."

$$0.99 < 1$$

"And so on and so on."

$$\begin{array}{rcl} 0.9 & < & 1 \\ 0.99 & < & 1 \\ 0.999 & < & 1 \\ 0.9999 & < & 1 \\ 0.99999 & < & 1 \\ & \vdots & \end{array}$$

"So doesn't that show that 0.999 is less than 1?"

$$0.999\cdots < 1 \quad ?$$

Yuri put down her pencil and shrugged.

"Seems just as right as the first one, anyway. But one of them must be wrong."

"An interesting dilemma," I said. "On the one hand, it seems like we can say $0.9,\ 0.99,\ 0.999\cdots$ gets as close to 1 as you like. On the other, it seems you can say that it never gets there."

"Well?" Yuri said, looking up at me with pleading eyes. "Which is it?"

4.1.5 My Explanation

I turned to a new page in the notebook, flattening the crease as I gathered my thoughts.

"Okay," I said. "Let me start with something like what you just said, but written out a little bit different. We'll start with a sequence like this."

$$\begin{aligned}
a_1 &= 0.9 \\
a_2 &= 0.99 \\
a_3 &= 0.999 \\
a_4 &= 0.9999 \\
a_5 &= 0.99999 \\
a_6 &= 0.999999 \\
&\vdots \\
a_n &= 0.\underbrace{9999\cdots9}_{n\ 9\text{s}} \\
&\vdots
\end{aligned}$$

Yuri nodded. "So you're naming them all as a with a subscript. Got it."

"Also, the subscript shows how many 9's there are. So that would seem to present this dilemma."

(1) The more 9's there are, the closer a_n is to 1.

(2) No matter how big n is, a_n is less than 1.

"Yes, exactly," Yuri said. "One of those has to be wrong. Right?"

"Well, no. These are both true statements."

"Huh? So $0.999\cdots$ is less than 1 after all? But I thought you said—"

"No, that's not right, either. $0.999\cdots = 1$ is a true statement, too."

"I am *so* confused," Yuri said.

She frowned as she concentrated on what we'd written. I left her alone to give her some time to ponder it all. There are few things more important to doing math than time to think. I thought about changing my slogan to 'Silence is the key to—'

A loud clatter of pans came from the kitchen. Apparently my mother was cooking again.

Yuri perked up with a grin.

"I've got it! We just have to change the definition of 'equals,' so that teeny tiny differences don't matter! Mathematicians change definitions all the time, right?"

It took me a moment to recover from the audacity of her suggestion.

"That's an amazing answer, Yuri. So amazing that I wish it were true. But we can't change what equals means. It really does mean absolutely, positively, right-on-the-nose the same."

Yuri's face fell.

"I give up, then. I don't get it."

Just then, a wail of despair came from the kitchen. Yuri and I ran downstairs to see what had happened.

"What's wrong?" I asked my mother, who was standing in front of the open refrigerator in an apron. She turned to me, pale and wide-eyed.

"We're out of eggs! I'd forgotten that I used them all last night!"

"Seriously? That's *it*?"

"But—but I was going to make omelettes!"

"Can't you make something else?"

"I've already got everything else prepared! I . . . " Mother's expression softened, and the tone in her voice became cloyingly sweet. "Honey, I know you're studying and all, but I don't suppose—"

"Awww, mom. It's freezing out there!"

"C'mon," Yuri said, laughing, "I'll go with you."

4.2 At the Supermarket

4.2.1 Arriving at Your Destination

We rode my bicycle to the supermarket, Yuri standing on the rear hub. I was thankful for the exercise to keep warm—it was even colder than I'd feared.

After a short search we found the eggs, paid at the register, and started back. Just as we were heading for the door, Yuri snagged my arm.

"Ooh, look what they've got!" She was pointing at the in-store snack counter. "Ice cream!"

"We've got to get back, Yuri. Mom's waiting for the eggs. Besides, ice cream in this weather? Are you insane?"

"Ice cream is one of those delicious-all-year-round foods," she said, maneuvering in front of me to cut off my escape route.

"Pleeease?" she said, making sad puppy eyes.

I sighed in resignation. We bought two vanilla cones and sat at the counter to eat them.

"Happy now?" I asked.

"Perfectly," Yuri said with a huge smile.

"At least one of us is," I grumbled. "And at least you aren't raging like you were before."

"I have no idea what you're talking about. Delicate flowers do not 'rage.' "

We sat there, licking our ice cream for a while.

"So what do you want to be when you grow up?" Yuri asked.

"I dunno. How about you?"

"Hmm . . . A lawyer, maybe."

"Sounds like you've been watching too many crime dramas lately."

"That's—that's possible. It would still be pretty cool, though."

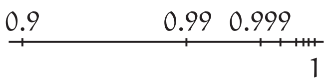
Yuri turned her cone in her hand, checking for wayward rivulets of ice cream. I pulled a mechanical pencil out of my breast pocket and looked around for something to write on. I settled on the back of a nearby store flier.

"Would it bug you if your wife made more money than you?" Yuri asked.

"Huh? Where did that come from?" I said, sketching a graph.

"Just wondering. I guess you wouldn't care. You're too oblivious to what people think."

"Whatever. Here, check this out," I said, pushing the graph toward her. "This is what we were talking about before."



"I know," Yuri said.

"See how this sequence $0.9, 0.99, 0.999$ is getting arbitrarily close to 1? It can get as close as you want it to be, because its destination is $0.999\cdots$."

"I said, *I know.* I understand that the sequence can get as close to 1 as you want it to. I get it. Done."

"So what's the problem?"

"That *closer* ain't *is.* Your explanations still aren't showing me how $0.999\cdots$ and 1 can be the same."

Glowering, Yuri took a joyless lick at her ice cream.

"Okay, how about this. I'm going to ask you some questions, you answer yes or no."

"Fine."

"If you continue the sequence $0.9, 0.99, 0.999$ on and on, will you eventually reach 1?"

"No. It doesn't matter how many nines you add, you still aren't there."

I nodded. "That's correct."

Yuri made a scary growling noise.

"You're *trying* to making me mad, aren't you."

"Hang on, hang on. I'm not done yet. One more question: as you continue the sequence $0.9, 0.99, 0.999$, is there some number that you're getting closer and closer to?"

"Yes, you're getting closer to 1. We've been through this like a million times."

"Right, but let me add one more thing. A notational rule, a way we use symbols to represent an idea."

"Okay, what's the rule?"

"That when we have a sequence like $0.9, 0.99, 0.999$, one that can get arbitrarily close to some number, we represent that 'some number' like this."

$$0.999\cdots$$

Yuri froze mid-lick, her eyes wide, staring at what I'd written.

She held up a hand and said, "Hold up a minute. So this is . . . "

I waited while she did some internal processing. Harsh winter sunlight gleamed gold in her hair.

"Okay," she said, "I think I finally get this. Let me make sure."

"Absolutely."

"This $0.999\cdots$, it's a representation of *some number*."

"Right."

"And that *some number* that it's representing is the number that it's getting closer and closer to."

"That it gets arbitrarily close to, yes."

"Arbitrarily close to, then. But because it never gets there, that *some number* never shows up."

"That's right."

"And that *some number* that $0.999\cdots$ represents is none other than 1."

"Exactly."

Yuri sighed.

"Okay, not only do I understand this now, I now understand what I didn't understand."

"Namely?"

"That $0.999\cdots$ isn't a number. Not the kind I'm used to, at least. It's a representation of a number."

Yuri jotted down some notes, pausing to catch escaping streams of melting ice cream as she did.

- $0.999\cdots$ represents *some number*.
- Continuing the sequence $0.9, 0.99, 0.999$, we can get arbitrarily close to that *some number*.
- But we never reach that *some number*, so it doesn't show up in the representation.
- The *some number* that $0.999\cdots$ represents is equal to 1

Yuri worked her way down the length of the cone as she reviewed this. When the cone had vanished, she gave a firm nod.

"This '$0.999\cdots$' has got to go. It's confusing," she said.

"How so?"

"Well think about it. Say you're writing a sequence of numbers like this."

$$0.9,\ 0.99,\ 0.999,\ \ldots$$

"You add those three dots at the end to mean 'goes on forever,' yeah? That made me think that when we were playing with $0.9, 0.99, 0.999$, a $0.999\cdots$ would eventually show up. But it doesn't, does it? There's no $0.999\cdots$ at the end of $0.9, 0.99, 0.999$ somewhere. We need a new way of writing this. Something like ..."

- $0.9, 0.99, 0.999, \ldots$ becomes arbitrarily close to $\heartsuit$.
- $\heartsuit$ is therefore equal to 1.

"This would make things so much clearer."

I nodded. "I agree."

"But it's *your. Fault. Too*," Yuri said, punctuating her words with finger jabs at my chest. "You should have told me from the beginning that $0.999\cdots$ was a notational whatever, not a normal number. This problem isn't really about math, it's just about how you write things!"

"Yeah, I guess you're right. Glad to see you get it now, though."

"Well at least I didn't see this at school first. There's no way I'd have figured out what was going on. That $0.999\cdots$ isn't some number that's eventually going to show up in a sequence, it's the destination where the sequence is heading, even if it never gets there.

And *that's* why $0.999\cdots$ and 1 are absolutely, positively, right-on-the-nose the same."

I nodded again, smiling.

"Oh," Yuri said, "I just realized something else. These two numbers are different too, right?"

$$0.999\cdots \qquad \text{(equal to 1)}$$
$$0.999\cdots9 \qquad \text{(less than 1)}$$

"They are. The three dots at the end of a number show where the number is heading. The three dots in the middle of a number is just an abbreviation. The first one never shows up in the sequence, the second one will, eventually. They're completely different things."

"They're way confusing things, is what they are."

"I know you'll never confuse them again, though."

I looked down, and saw a white plastic bag at my feet. It took me a moment to realize what I was looking at.

"The eggs! Mom's still waiting for us!"

Answer 5-1

The following is a true statement.

$$0.999\cdots = 1$$

4.3 In the Music Room

4.3.1 Introducing Variables

"You think it was Yuri's boyfriend that gave her that problem?" Tetra whispered.

We were hanging out in the music room after classes. Seated next to Miruka at the piano was her friend Ay-Ay, president of the school piano club "Fortissimo." Ay-Ay was in eleventh grade, like Miruka and me, but in a different homeroom. She was into music like I was into math, and spent most of her free time here, practicing. The music room was normally locked up after hours, but she was so talented that the music department head had given her a key.

We watched Miruka and Ay-Ay's backs as they played, Miruka's long, straight hair in sharp juxtaposition to Ay-Ay's wavy locks. They took turns playing pieces and argued between each. Ay-Ay would insist that they differentiate between 'mechanical Bach' and 'celestial Bach,' while Miruka argued the need to extract the 'formal Bach' from the 'meta-Bach.'

I had not a single clue what they were talking about.

"No way," I whispered back to Tetra, knowing that Miruka was in no mood to have her music interrupted.

"Betcha he is," Tetra said, a sly smile on her face. "Or wants to be. He's trying to impress her. It's a sign of affection."

I brushed away her words with a gesture.

"Yuri sure is smart," Tetra continued, a hint of admiration in her voice. "I still can't shake the feeling that $0.999\cdots$ is a little bit less than 1."

Tetra pulled her notebook and a pencil out of her bag.

"You said that when you explained all that to Yuri, you used n nines in the decimal. Like this, right?"

$$a_n = 0.\underbrace{999\cdots9}_{n \text{ 9s}}$$

"Sure. Using variables like that can make explanations a lot clearer. In this case two variables, I guess—the a and its subscript n. Anyway, doing that's a lot easier than saying 'the number of nines in the number $0.999\cdots9$.'"

"I can see that. I need to get a lot more comfortable with introducing variables like this. It doesn't come natural to me at all. Seeing more letters still just makes everything look more confusing."

Tetra began writing letters in her notebook, as if practicing.

I looked back to the piano, where it was Ay-Ay's turn to play. Miruka had gotten up and was standing behind Ay-Ay with her arms crossed. She glanced back at me but immediately returned her gaze to the keys.

4.3.2 Limits

I motioned Tetra back to a further corner of the music room, where we could speak in more normal voices.

"Let's talk about limits, then," I said. I held out my hand, and Tetra passed me the notebook and pencil.

"Say you have some number a_n, and the bigger you make n, the closer a_n gets to some certain number. In a situation like that, the number you're getting close to is called a limit. You write it like this."

$$\lim_{n \to \infty} a_n$$

"Let's use A to name the number that a_n is getting closer and closer to. Then you can say that this limit equals A, which you write like this."

$$\lim_{n \to \infty} a_n = A$$

"You can also write it without using the limit notation, like this."

$$a_n \to A \qquad \text{as} \qquad n \to \infty$$

"A couple more words to learn. When a sequence can get arbitrarily close to some number, you say it *converges* to that number. So saying that a sequence converges is the same as saying it has a limit. Also, finding the limit of some sequence is often called *taking* the limit."

The limit of a sequence

$$\begin{aligned} & a_n \text{ gets arbitrarily close to } A \text{ as } n \text{ increases} \\ \iff & \lim_{n \to \infty} a_n = A \\ \iff & a_n \to A \quad \text{as} \quad n \to \infty \\ \iff & \text{sequence } \langle\, a_n \,\rangle \text{ converges to } A \end{aligned}$$

I watched Tetra's eyes as she read what I'd written. She pointed at a line.

"How do you read this?"

$$a_n \to A \qquad \text{as} \qquad n \to \infty$$

"You'd say 'a_n approaches A as n approaches infinity.'"

"And this one?"

$$\lim_{n\to\infty} a_n = A$$

"I'd read that, 'the limit of a_n as n approaches infinity is A.'"

"Hmm... I think the first one is easier to understand, but the second one is the right one, right?"

"It isn't more correct, but it's what you'll see most often. It's more compact, if nothing else."

Tetra reflected on this for a moment, her eyebrows drawing in.

"Since we're talking about things getting closer, why the equals sign? Shouldn't it be this?"

$$\lim_{n\to\infty} a_n \to A \quad ?$$

"Ah, interesting," I said. "But no, the arrow shows change. We want to use it in the $n \to \infty$ part, to show that n is getting bigger and bigger. But if we wrote $\lim_{n\to\infty} a_n \to A$, that would mean the limit of a_n is getting closer to A."

"Isn't it?"

"No. Watch out for that. The limit $\lim_{n\to\infty} a_n$ is a specific number, the number that $\langle a_n \rangle$ converges to. That doesn't change."

$$\begin{array}{lcll} \lim_{n\to\infty} a_n & \to & A & \text{incorrect} \\ \lim_{n\to\infty} a_n & = & A & \text{correct} \end{array}$$

"Oops, sorry, I've got it straight now." Tetra said, blushing. She looked back at the notebook. "Oh, one more thing. Not all sequences will converge to something, right?"

"Absolutely not. What does a sequence like this do?"

$$10, 100, 1000, 10000, \ldots$$

"It keeps getting bigger and bigger and bigger." Tetra spread her arms wider and wider to illustrate.

"That's right, and it never stops getting bigger. There's no way it will get closer and closer to some specific number. We say a sequence like this doesn't *con*verge, it *di*verges. This particular sequence never stops getting bigger, so we can say that it diverges to positive infinity."

"We can't say that it *converges to* positive infinity?"

"No. Positive infinity isn't a number, so you can't get arbitrarily close to it. So you can never say that a sequence has infinity as a limit, or that it converges to infinity. You can just say that it diverges to positive infinity—or negative infinity, if it's heading that way."

"Okay, got it."

4.3.3 Sound Makes the Music

I heard voices coming toward us.

"I'm telling you, the C♯ just doesn't work there," Ay-Ay was saying.

"Oh, I don't know . . . " Miruka said.

"It breaks the pattern!"

"Maybe that's why my fingers don't seem to want to hit it."

The two walked up to where we had taken refuge. Ay-Ay's face was dark.

"Taking a break?" I asked.

"What are you two talking about?" Miruka said, ignoring me.

"Limits!" Tetra said. "This is some tricky stuff."

"Oh?" Miruka said, cocking her head.

"Well, I think I get the idea of getting arbitrarily close to something, but when you put it all in symbols I'm not so sure any more. I lose my instinct for it, somehow."

Ay-Ay stepped forward and inserted herself into the conversation.

"Listen," she said, "I'm no mathematician, but the way I see it, you use all those symbols in math because that's the best way to say what needs to be said."

Ay-Ay held out her hands and examined her palms. She turned them over and considered their backs for a time. I regarded her impressively long, strong-looking fingers—the fingers of a true pianist.

"Music is all about sounds," she continued, still looking at her hands. "Just . . . sounds." Her tone was uncharacteristically serious.

"Sometimes you can describe the world using words. When you can do that, great. Have at it. But there's also a world that can only be described with sound."

She formed a hand into a fist, thumb extended, and used that to point to her chest.

"Music is *mine*. It's what I use to let loose feelings that would rip me to shreds if I kept them pent up. It's all I *can* use. So I eat for music, I breathe for music, I live for music."

Ay-Ay's solemn air and grim expression left no room for response.

"Sometimes I meet people who say they 'don't get' music," she said. "They're usually the kind of person who can't 'get' anything they can't put into words, because music demands understanding on its own terms. Music isn't about words, it's about sounds. It can only *be* sounds. If you think you can express it in words, you aren't really listening. If you're searching for words, you aren't experiencing the performance, you aren't hearing the music. You're in a different time and space, where the music isn't. I want to scream at people like that, 'Stop looking for words! Open your ears!'"

Ay-Ay paused, took a deep breath, and looked at Tetra.

"If you try to study math without reading the equations, without *really* reading them, aren't you doing the same thing?"

Tetra let out a small gasp.

"Wow, I totally see what you mean," Tetra said. "If you don't read equations, you aren't seeing the world that mathematics is presenting. I guess if you try to stick to words without embracing the math, you aren't really *doing* math, you're just kind of watching it."

"Music and mathematics seem like completely different things on the surface," I said, "but in a lot of ways they're really similar."

Ay-Ay nodded.

"Musicians describe a world of music, so shut up and listen to the sounds. Mathematicians describe a world of mathematics, so shut up and embrace the equations."

Tetra smiled. "So sounds are the words of music, and equations are the words of mathematics!"

"It's always back to words with this girl," Ay-Ay grumbled.

"Oh, not literally!" Tetra rushed to add. "I just mean . . . as a basic unit of representation."

"It's not only equations, though," I said. "It's concepts, too. Like, when we talk about limits, we say a value gets 'arbitrarily close to' some other number, not that it reaches it. I think a deep understanding of concepts like that are vital to understanding the equations that represent them."

"In any case," Ay-Ay said, "I'll stick to my music. I don't know if I'll be able to get a job related to music after I graduate, but it's something that will be part of me for the rest of my life. Absolutely."

Ay-Ay slapped her hands together, shattering the somber mood she'd created.

"Why're you guys all acting so *serious*?" she said, laughing. "Time to lighten up."

I smiled at her.

"I don't think you have anything to worry about," I said, "about working with music and all. Your playing is amazing, and your compositions are genius. I can't wait to see what you go on to do."

Ay-Ay pounded me on the back.

"You're a good kid," she said. "For a math nerd."

4.3.4 Calculating Limits

Ay-Ay left the room, saying she needed a break. The purely musical element of our set now gone, our talk soon turned to math.

"So did you show Tetra how to take some basic limits?" Miruka asked me.

"Like what?" I asked.

"Like this," she said, helping herself to my notebook and pencil.

Problem 5-2 (Basic limits)
$\lim_{n\to\infty} \frac{1}{10^n}$

"Uh, well . . . no?" Tetra said, turning panicky eyes my way.

"Give me a shot at this?" I offered.

"Be my guest," Miruka said, smiling as she handed back the pencil and paper.

"Okay, Tetra. So what we want to find is the value of this expression."

$$\lim_{n \to \infty} \frac{1}{10^n}$$

Tetra nodded. "So we're looking for the limit of $\frac{1}{10^n}$ as n heads off toward infinity, right?"

"Right. In terms like you suggested before, we're looking for the value of the 'club' in this expression."

$$\frac{1}{10^n} \to \clubsuit \quad \text{as} \quad n \to \infty$$

"Got it! How do we start?"

"As always, with an explicit representation of this sequence. Like I always say, examples are the key to understanding. Anyway, the sequence looks like this."

$$\frac{1}{10^1}, \frac{1}{10^2}, \frac{1}{10^3}, \frac{1}{10^4}, \frac{1}{10^5}, \ldots, \frac{1}{10^n}, \ldots$$

"What we want to know," I continued, "is if $\frac{1}{10^n}$ is heading for some specific value as n gets bigger and bigger. And if it is, we want to know what that number is."

"Hmm . . ." Tetra said, her eyebrows knitting in concentration.

"Let me give you a hint—for now, just pay attention to the denominators."

$$10^1,\ 10^2,\ 10^3,\ 10^4,\ 10^5,\ \ldots,\ 10^n,\ \ldots$$

"Oh," Tetra said. "So they're increasing like this?"

$$10,\ 100,\ 1000,\ 10000,\ 100000,\ \ldots,\ 10^n,\ \ldots$$

"That's right. So as n gets bigger, what happens to 10^n?"

"It gets bigger, too. Like, *way* bigger."

"Exactly. That means we can say this."

$$10^n \to \infty \quad \text{as} \quad n \to \infty$$

I tapped what I'd written with my pencil.

"This means that as n gets bigger, the denominator of $\frac{1}{10^n}$ gets bigger, without limit. So what will happen to the fraction $\frac{1}{10^n}$ as its denominator keeps increasing?"

"The fraction . . . should get smaller and smaller, right?"

"Exactly. In fact, we can make it arbitrarily close to 0, just by making n big enough. That means we can say this."

$$\frac{1}{10^n} \to 0 \quad \text{as} \quad n \to \infty$$

"Putting this into the standard form for limits, we get this."

$$\lim_{n\to\infty} \frac{1}{10^n} = 0$$

"So we've shown that the limit exists, and its value is 0."

Answer 5-2 (Basic limits)

$10^n \to \infty$ as $n \to \infty$, so $\frac{1}{10^n} \to 0$. We therefore have that

$$\lim_{n\to\infty} \frac{1}{10^n} = 0.$$

"Huh . . ." Tetra said, chewing on a nail as she pondered this. "One question?" she said.

"You bet."

"This came up in how you explained the problem, right?"

$$10^n \to \infty \quad \text{as} \quad n \to \infty$$

"It did."

"And does that mean we can write this?"

$$\lim_{n\to\infty} 10^n = \infty$$

"Yep. Something wrong with that?"

"Well, it's just that—maybe I'm not understanding this right, but doesn't this mean you're saying that the limit of 10^n here is infinity?"

"Sure."

"But didn't you also say that we can't say the limit of a sequence is infinity? Just that it diverges to infinity?"

"Ah, right. Sorry, I didn't explain that well enough. You're right to be suspicious here, since infinity isn't a number. But what's going on is that we've expanded the definition of the = operator to mean this."

$$\lim_{n\to\infty} 10^n = \infty \quad\Longleftrightarrow\quad 10^n \to \infty \quad \text{as} \quad n \to \infty$$

"Oh, okay" Tetra said. "So $\lim_{n\to\infty} 10^n = \infty$ is another way of saying that the sequence $\langle 10^n \rangle$ diverges to infinity."

I nodded. "Yes, that's right."

An impatient Miruka broke her silence.

"Next problem."

Problem 5-3 (basic limits)

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{10^k}$$

Tetra looked back and forth between this problem and the previous one.

"Um . . . is this not the same thing as what we just did?" she asked.

Miruka grimaced. "Who was it that was just going on about how not reading equations prevents you from seeing the world that mathematics is presenting?"

"Okay, let me read this one more time, carefully." Tetra pulled the notebook closer and peered at the problem. "Oh, I get it," she said. "The sigma makes all the difference, doesn't it. I have no idea how to solve this, though. How do you take the limit of a sum?"

"Have fun," Miruka said, patting me on the shoulder. She turned and headed back to the piano.

"Okay," I said. "Here's what we want to solve."

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{10^k}$$

"To do that, first we need to pay attention to exactly what it is we're taking the limit of."

$$\sum_{k=1}^{n} \frac{1}{10^k}$$

"Let's think about how we can represent this as an expression involving just n. That'll be a lot easier than dealing with the sigma. So what's the first thing to do to make sure you understand what's going on here?"

"I know! Write out some examples!"

Tetra took the pencil and started writing.

$$\sum_{k=1}^{1} \frac{1}{10^k} = \frac{1}{10^1} \qquad \text{(for } n = 1\text{)}$$

$$\sum_{k=1}^{2} \frac{1}{10^k} = \frac{1}{10^1} + \frac{1}{10^2} \qquad \text{(for } n = 2\text{)}$$

$$\sum_{k=1}^{3} \frac{1}{10^k} = \frac{1}{10^1} + \frac{1}{10^2} + \frac{1}{10^3} \qquad \text{(for } n = 3\text{)}$$

"Very good," I said. "Can you use this to write a general expression?"

"I think so . . . Yeah, I can!"

$$\sum_{k=1}^{n} \frac{1}{10^k} = \frac{1}{10^1} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots + \frac{1}{10^n} \qquad \text{(general expression)}$$

"Good job—now we've established the groundwork for finding this limit. Next we want to massage this into a more useful form, one with the terms shifted. We can do that by multiplying both sides of the equation by $\frac{1}{10}$."

$$\sum_{k=1}^{n} \frac{1}{10^k} = \frac{1}{10^1} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots + \frac{1}{10^n} \qquad \text{general expression}$$

$$\frac{1}{10} \cdot \sum_{k=1}^{n} \frac{1}{10^k} = \frac{1}{10} \cdot \left(\frac{1}{10^1} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots + \frac{1}{10^n} \right) \qquad \text{mult. both sides by } \frac{1}{10}$$

$$\frac{1}{10} \cdot \sum_{k=1}^{n} \frac{1}{10^k} = \frac{1}{10} \cdot \frac{1}{10^1} + \frac{1}{10} \cdot \frac{1}{10^2} + \frac{1}{10} \cdot \frac{1}{10^3} + \cdots + \frac{1}{10} \cdot \frac{1}{10^n} \qquad \text{expand the right side}$$

$$\frac{1}{10} \cdot \sum_{k=1}^{n} \frac{1}{10^k} = \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \cdots + \frac{1}{10^{n+1}} \qquad \text{expr. with terms shifted}$$

Tetra ran a finger down each line, confirming what I was doing in each step.

"By 'shifted terms,' you mean that the exponents on the 10s are each increased by 1, right?"

"That's right," I said. "Now we can subtract this shifted equation from the generalized equation. That will kill off all the intermediate terms."

$$\begin{array}{rrcllllll}
 & \sum_{k=1}^{n} \frac{1}{10^k} & = & \frac{1}{10^1} + & \frac{1}{10^2} + & \frac{1}{10^3} + \cdots + & \frac{1}{10^n} & & \text{general expr.} \\
- & \frac{1}{10} \cdot \sum_{k=1}^{n} \frac{1}{10^k} & = & & \frac{1}{10^2} + & \frac{1}{10^3} + \cdots + & \frac{1}{10^n} + & \frac{1}{10^{n+1}} & \text{shifted expr.} \\
\hline
 & \left(1 - \frac{1}{10}\right) \cdot \sum_{k=1}^{n} \frac{1}{10^k} & = & \frac{1}{10^1} & & & & - \frac{1}{10^{n+1}} & \text{difference}
\end{array}$$

"Oh, neat!" Tetra said. "Everything except the first and last terms went away!"

"Let's calculate this and see what we get."

$$\begin{aligned}
\left(1 - \frac{1}{10}\right) \cdot \sum_{k=1}^{n} \frac{1}{10^k} &= \frac{1}{10^1} - \frac{1}{10^{n+1}} && \text{previous equation} \\
\frac{10-1}{10} \cdot \sum_{k=1}^{n} \frac{1}{10^k} &= \frac{1}{10^1} - \frac{1}{10^{n+1}} && \text{calculate left side} \\
\frac{9}{10} \cdot \sum_{k=1}^{n} \frac{1}{10^k} &= \frac{1}{10^1} - \frac{1}{10^{n+1}} && \text{simplify left side} \\
\sum_{k=1}^{n} \frac{1}{10^k} &= \left(\frac{1}{10^1} - \frac{1}{10^{n+1}}\right) \cdot \frac{10}{9} && \text{multiply both sides by } \frac{10}{9} \\
&= \frac{1}{10^1} \cdot \frac{10}{9} - \frac{1}{10^{n+1}} \cdot \frac{10}{9} && \text{distribute} \\
&= \frac{1}{9} - \frac{1}{9 \cdot 10^n} && \text{simplify}
\end{aligned}$$

I rechecked my work. Satisfied, I nodded.

"Now we need to think about what's going to happen to the right side here as n goes to infinity."

$$\sum_{k=1}^{n} \frac{1}{10^k} = \frac{1}{9} - \frac{1}{9 \cdot 10^n}$$

"Hmm," Tetra said, putting a finger to her lips. "When n goes to infinity, the limit of the $\frac{1}{9 \cdot 10^n}$ part here will be 0, won't it?"

"It will. That means we can say this."

$$\sum_{k=1}^{n} \frac{1}{10^k} \to \frac{1}{9} \quad \text{as} \quad n \to \infty$$

"In other words..."

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{10^k} = \frac{1}{9}$$

Answer 5-3 (basic limits)

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{10^k} = \frac{1}{9}$$

"You guys done?"

I glanced back and saw Miruka standing behind us, holding some sheet music. I nodded.

"Calculate $0.999\cdots$ next, then."

Problem 5-4

Calculate $0.999\cdots$, defining $0.999\cdots$ as follows:

$$0.999\cdots = \lim_{n\to\infty} 0.\underbrace{999\cdots9}_{n\text{ 9s}}$$

I blinked at this unexpected problem, but soon laughed.

"So *this* is what you've been leading us to."

"You finally noticed," she said, taking the notebook and solving the problem herself.

$$
\begin{aligned}
0.999\cdots &= \lim_{n\to\infty} 0.\underbrace{999\cdots9}_{n\text{ 9's}} \\
&= \lim_{n\to\infty} \left(0.9 + 0.09 + 0.009 + \cdots + 0.\underbrace{000\cdots0}_{(n-1)\text{ 0's}}9\right) \\
&= \lim_{n\to\infty} \left(\frac{9}{10^1} + \frac{9}{10^2} + \frac{9}{10^3} + \cdots + \frac{9}{10^n}\right) \\
&= \lim_{n\to\infty} 9 \cdot \left(\frac{1}{10^1} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots + \frac{1}{10^n}\right) \\
&= \lim_{n\to\infty} 9 \cdot \sum_{k=1}^{n} \frac{1}{10^k} \\
&= 9 \cdot \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{10^k} \\
&= 9 \cdot \frac{1}{9} \qquad \text{from Answer 5-3}
\end{aligned}
$$

"And thus, $0.999\cdots$ equals 1," Miruka said.

Answer 5-4

$$0.999\cdots = \lim_{n\to\infty} 0.\underbrace{999\cdots9}_{n\text{ 9s}} = 1$$

"Wow," Tetra said. "So you can just, like, calculate $0.999\cdots$ out?"

"Yeah," I said, "if you're clever about how you define things."

"Infinity fools the senses," Miruka said. "If you try to rely on common sense when you deal with infinity, you'll get tripped up every time. Not all of us can be an Euler."

"I see," Tetra said.

"So don't rely on your senses, rely on—" Miruka looked at me.

"Logic," I said.

Miruka turned to Tetra.

"Don't rely on words, rely on—"

"Equations."

Miruka smiled, and Tetra raised her hand.

"And that's why we use this lim operator, instead of words like 'gets closer and closer to' and all, right?"

Miruka gave a reluctant nod.

"Yes, but the way we've been treating limits so far, the lim operator isn't much better than just a word."

"Why not?"

"Because we haven't defined it. Not mathematically, at least." Miruka began walking slowly around us. "At some point we're going to have to leave the words behind."

"But . . . but how?" Tetra asked.

"By using equations instead, of course," Miruka replied.

"You can use equations to *define* limits? Not just find them?"

"Now we can," Miruka replied with a grin. "But actually that's a relatively new development. Cauchy first brought rigorous concepts of limits into mathematics in the early 1800s, but it wasn't until late in that century that Weierstrass finally gave us a full definition using equations."

The door creaked as Ay-Ay reentered the room.

"Break's over!" she announced. "Miruka! Back at it!"

Tetra was still mumbling to herself, a perplexed look on her face.

"Limits . . . ? With equations . . . ?"

Miruka playfully bopped Tetra's head as she headed back toward the piano.

"We'll get there soon," she said, smiling. "To the realm of epsilon–delta."

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Other works by Hiroshi Yuki

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